Natural Language Understanding

Lecture 2: Revision of neural networks and backpropagation

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Biological neural networks

- Neuron receives inputs and combines these in the cell body.
- If the input reaches a threshold, then the neuron may fire (produce an output).
- Some inputs are excitatory, while others are inhibitory.
The relationship of artificial neural networks to the brain

While the brain metaphor is sexy and intriguing, it is also distracting and cumbersome to manipulate mathematically. (Goldberg 2015)
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The perceptron: an artificial neuron

Developed by Frank Rosenblatt in 1957.
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Input function:
\[ u(x) = \sum_{i=1}^{n} w_i x_i \]
The perceptron: an artificial neuron

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Activation function: threshold
\[ y = f(u(x)) = \begin{cases} 
1, & \text{if } u(x) > \theta \\
0, & \text{otherwise} 
\end{cases} \]
The perceptron: an artificial neuron

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Input function:
\[ u(x) = \sum_{i=1}^{n} w_i x_i \]

Activation function: threshold
\[ y = f(u(x)) = \begin{cases} 1, & \text{if } u(x) > \theta \\ 0, & \text{otherwise} \end{cases} \]

Activation state:
0 or 1 (-1 or 1)
The perceptron: an artificial neuron

Developed by Frank Rosenblatt in 1957.

\[
\sum f(x_1, x_2, \ldots, x_n) \cdot (w_1, w_2, \ldots, w_n) = y
\]

- Inputs are in the range \([0, 1]\), where 0 is “off” and 1 is “on”.
- Weights can be any real number (positive or negative).
Perceptrons can represent logic functions

**Perceptron for AND**

\[ f \text{ if } \sum x_i \geq 1 \text{ then } 1 \text{ else } 0 \]

\[ f(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 + x_2 \geq 1 \\ 0 & \text{otherwise} \end{cases} \]

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Perceptrons can represent logic functions

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Perceptrons can represent logic functions.

**Perceptron for AND**

\[ \text{if } \sum \geq 1 \text{ then } 1 \text{ else } 0 \]

0 \cdot 0.5 + 1 \cdot 0.5 = 0.5 < 1

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Perceptrons can represent logic functions

**Perceptron for AND**

\[ f \text{ if } \sum x \geq 1 \text{ then } 1 \text{ else } 0 \]

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 1 & 1 \\
\end{array}
\]

\[ x_1 \quad x_2 \quad x_1 \text{ AND } x_2 \]
Perceptrons can represent logic functions

Perceptron for AND

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**Perceptron for AND**

- \( f \) if \( \sum \geq 1 \) then 1 else 0

\[
1 \cdot 0.5 + 1 \cdot 0.5 = 1 = 1
\]

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Perceptrons can represent logic functions

Perceptron for OR

\[ f \text{ if } \sum \geq 0.5 \text{ then } 1 \text{ else } 0 \]

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Perceptron for OR

\[ f \text{ if } \sum \geq 0.5 \text{ then } 1 \text{ else } 0 \]

\[ 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5 = 0.5 \]

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\[ f(x_1, x_2) = \begin{cases} 1 & \text{if } \sum x_i w_i \geq 0.5 \\ 0 & \text{else} \end{cases} \]

\[ 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5 = 0.5 \]

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The Perceptron for $\text{NOT(OR)}$ is represented by the function:

$$f(\sum_{i} x_i) \geq ???$$

where $f$ is a threshold function, and $x_i$ are the inputs. The table shows the correct outputs for the OR operation:

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Perceptrons are linear classifiers

\[ x = \sum_{i=0}^{n} w_i x_i \]
Perceptrons are linear classifiers, i.e., they can only separate points with a hyperplane (a straight line).
Perceptron can learn logic functions from examples

Give some examples to the Perceptron:

<table>
<thead>
<tr>
<th>N</th>
<th>input</th>
<th>target t</th>
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<tbody>
<tr>
<td>1</td>
<td>(0,1,0,0)</td>
<td>1</td>
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<td>2</td>
<td>(1,0,0,0)</td>
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<td>(1,0,1,0)</td>
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- Input: a vector of 1’s and 0’s—a feature vector.
- Output: a 1 or 0, given as the target.
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<th>output ( o )</th>
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- Input: a vector of 1’s and 0’s—an **feature vector**.
- Output: a 1 or 0, given as the target.
- How do we efficiently find the weights and threshold?
Q₁: Choosing weights and threshold $\theta$ for the perceptron is not easy! What’s an effective to learn the weights and threshold from examples?

A₁: We use a learning algorithm that adjusts the weights and threshold based on examples.

http://www.youtube.com/watch?v=vGwemZhPlsA&feature=youtu.be
Simplify by converting $\theta$ into a weight

$$\sum_{i=1}^{n} w_i x_i > \theta$$
Simplify by converting $\theta$ into a weight

$$\sum_{i=1}^{n} w_i x_i > \theta$$

$$\sum_{i=1}^{n} w_i x_i - \theta > 0$$
Simplify by converting $\theta$ into a weight

\[ \sum_{i=1}^{n} w_i x_i > \theta \]
\[ \sum_{i=1}^{n} w_i x_i - \theta > 0 \]

\[ w_1 x_1 + w_2 x_2 + \ldots + w_n x_n - \theta > 0 \]
Simplify by converting \( \theta \) into a weight

\[
\sum_{i=1}^{n} w_i x_i > \theta
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\[
\sum_{i=1}^{n} w_i x_i - \theta > 0
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\[
w_1 x_1 + w_2 x_2 + \ldots w_n x_n - \theta > 0
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\[
w_1 x_1 + w_2 x_2 + \ldots w_n x_n + \theta(-1) > 0
\]
Simplify by converting $\theta$ into a weight

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$$w_1 x_1 + w_2 x_2 + \ldots w_n x_n + \theta(-1) > 0$$
Simplify by converting $\theta$ into a weight

$x_0 = -1$

Let $x_0 = -1$ be the weight of $\theta$. Now our activation function is:

$$y = f(u(x)) = \begin{cases} 1, & \text{if } u(x) > 0 \\ 0, & \text{otherwise} \end{cases}$$
Learn by adjusting weights whenever output $\neq$ target.

Intuition: classification depends on the sign (+ or -) of the output. If output has a different sign than the target, adjust weights to move output in the direction of 0.
Learn by adjusting weights whenever output $\neq$ target

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- $o = 0$ and $t = 0$  Don’t adjust weights
Learn by adjusting weights whenever output $\neq$ target

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- $o = 0$ and $t = 0$: Don’t adjust weights
- $o = 0$ and $t = 1$: $u(x)$ was too low. Make it bigger!
Learn by adjusting weights whenever output $\neq$ target

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- $o = 1$ and $t = 1$: Don’t adjust weights

Notice: the sign of $t - o$ is the direction we want to move in.
Learn by adjusting weights whenever output $\neq$ target

**Perceptron Learning Rule**

$$w_i \leftarrow w_i + \Delta w_i$$

$$\Delta w_i = \eta(t - o)x_i$$

- $\eta$, $0 < \eta \leq 1$ is a constant called the **learning rate**.
- $t$ is the target output of the current example.
- $o$ is the output of the Perceptron with the current weights.
Learning Rule

Perceptron Learning Rule

\[ w_i \leftarrow w_i + \Delta w_i \]

\[ \Delta w_i = \eta (t - o) x_i \]

- Learning rate \( \eta \) is positive; controls how big changes \( \Delta w_i \) are.
- If \( x_i > 0 \), \( \Delta w_i > 0 \). Then \( w_i \) increases in an so that \( w_i x_i \)
  becomes larger, increasing \( u(x) \).
- If \( x_i < 0 \), \( \Delta w_i < 0 \). Then \( w_i \) reduces so that the absolute
  value of \( w_i x_i \) becomes smaller, increasing \( u(x) \).
Perceptron Learning Rule

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- If \( x_i < 0 \), \( \Delta w_i < 0 \). Then \( w_i \) reduces so that the absolute value of \( w_i x_i \) becomes smaller, increasing \( u(x) \).
Learning Algorithm

1. Initialize all weights randomly.
2. repeat
3. for each training example do
4. Apply the learning rule.
5. end for
6. until the error is acceptable or a certain number of iterations is reached

This algorithm is guaranteed to find a solution with error zero in a limited number of iterations as long as the examples are linearly separable.
Perceptrons can represent some logic functions... but not all!

Perceptron for XOR

\[
f(\mathbf{x}) = \begin{cases} 
0 & \text{if } \sum x_i \geq \theta \\
1 & \text{else}
\end{cases}
\]

XOR is an exclusive OR because it only returns a true value of 1 if the two values are exclusive, i.e., they are both different.
Perceptrons can represent some logic functions... but not all!

Perceptron for XOR

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Perceptron for XOR

\[ f(x_1, x_2) = \begin{cases} 1 & \text{if } \sum x_i \geq \theta \\ 0 & \text{else} \end{cases} \]

\[ 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5 \]

XOR is an exclusive OR because it only returns a true value of 1 if the two values are exclusive, i.e., they are both different.

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<th>( x_1 )</th>
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\[ 1 \cdot 0.5 + 1 \cdot 0.5 = 1 \]

XOR is an **exclusive OR** because it only returns a **true** value of 1 if the two values are exclusive, i.e., they are both different.
Problem: XOR is not linearly separable
Multilayer Perceptrons (MLPs) are more expressive

- MLPs are **feed-forward** neural networks, organized in layers.
- One **input** layer, one or more **hidden** layers, one **output** layer.
- Each node in a layer connected to all other nodes in next layer.
- Each connection has a weight (can be zero).
- Universal function approximators: can represent XOR.
Q: How would you represent XOR?
We can use activation functions other than thresholds

\[ y = \sum w_i x_i \]

**Step function**

Outputs 0 or 1.

**Sigmoid function**

Outputs a real value between 0 and 1.
Sigmoid can be made sharper or smoother
1. Present the pattern at the input layer.
1. Present the pattern at the input layer.
2. Calculate activation of input neurons
1. Present the pattern at the input layer.
2. Calculate activation of input neurons
1. Present the pattern at the input layer.
2. Calculate activation of input neurons
Forward Pass

1. Present the pattern at the input layer.
2. Calculate activation of input neurons.
1. Present the pattern at the input layer.
2. Calculate activation of input neurons
1. Present the pattern at the input layer.
2. Calculate activation of input neurons
4. Read the network output from both output neurons.
Learning in multilayer perceptrons

**General Idea: same as in a simple perceptron**

1. Send the MLP an input pattern, $x$, from the **training set**.
2. Get the output from the MLP, $y$.
3. Compare $y$ with the “right answer”, or target $t$, to get the **error quantity**.
4. Use the error quantity to modify the weights, so next time $y$ will be closer to $t$.
5. Repeat with another $x$ from the training set.

When updating weights after seeing $x$, the network doesn’t just change the way it deals with $x$, but other inputs too . . . Inputs it has not seen yet!

**Generalization** is the ability to deal accurately with unseen inputs.
The perceptron learning rule minimizes the difference between the actual and desired outputs:

$$w_i \leftarrow w_i + \eta (t - o)x_i$$

**Generalization of this: Mean Squared Error (MSE)**

An *error function* represents such a difference over a set of inputs:

$$E(\vec{w}) = \frac{1}{2N} \sum_{p=1}^{N} (t^p - o^p)^2$$

- $N$ is the number of patterns
- $t^p$ is the target output for pattern $p$
- $o^p$ is the output obtained for pattern $p$
- the 2 makes little difference, but makes life easier later on!
Minimize error by gradient descent

Interpret $E$ just as a mathematical function depending on $\vec{w}$ and forget about its semantics, then we are faced with a problem of mathematical optimization.

$$\min_{\vec{u}} f(\vec{u})$$

We consider only continuous and differentiable functions.
Gradient and Derivatives: The Idea

- **Gradient descent** can be used for minimizing functions.
- The derivative is a measure of the rate of change of a function, as its input changes;
- For function \( y = f(x) \), the derivative \( \frac{dy}{dx} \) indicates how much \( y \) changes in response to changes in \( x \).
- If \( x \) and \( y \) are real numbers, and if the graph of \( y \) is plotted against \( x \), the derivative measures the slope or gradient of the line at each point, i.e., it describes the steepness or incline.
So, we know how to use derivatives to adjust one input value. But we have several weights to adjust! We need to use partial derivatives. A partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant.

**Example**

If \( y = f(x_1, x_2) \), then we can have \( \frac{\partial y}{\partial x_1} \) and \( \frac{\partial y}{\partial x_2} \).

Given partial derivatives, update the weights:

\[
\begin{align*}
    w_{ij}' &= w_{ij} + \Delta w_{ij} \\
    \text{where } \Delta w_{ij} &= -\eta \frac{\partial E}{\partial w_{ij}} .
\end{align*}
\]
Learning Rate
Learning Rate

Small $\eta$ leads to convergence.
Very small $\eta$, convergence may take very long.
Case of medium size $\eta$, also converges.
Very large $\eta$: divergence.
Pure gradient descent is a nice theoretical framework but of limited power in practice.

Finding the right $\eta$ is annoying. Approaching the minimum is time consuming.

Heuristics to overcome problems of gradient descent:

- gradient descent with momentum
- individual learning rates for each dimension
- adaptive learning rates
- decoupling step length from partial derivatives
Summary So Far

- We learnt what a multilayer perceptron is.
- We know a learning rule for updating weights in order to minimise the error:
  \[ w'_{ij} = w_{ij} + \Delta w_{ij} \]
  where \( \Delta w_{ij} = -\eta \frac{\partial E}{\partial w_{ij}} \)
- \( \Delta w_{ij} \) tells us in which direction and how much we should change each weight to roll down the slope (descend the gradient) of the error function \( E \).
- So, how do we calculate \( \frac{\partial E}{\partial w_{ij}} \)?
The mean squared error function $E$, which we want to minimize:

$$E(\vec{w}) = \frac{1}{2N} \sum_{p=1}^{N} (t^p - o^p)^2$$
If we use a sigmoid activation function $f$, then the output of neuron $i$ for pattern $p$ is:

$$o_i^p = f(u_i) = \frac{1}{1 + e^{au_i}}$$

where $a$ is a pre-defined constant and $u_i$ is the result of the input function in neuron $i$:

$$u_i = \sum_j w_{ij}x_{ij}$$
For the $p$th pattern and the $i$th neuron, we use gradient descent on the error function:

$$\Delta w_{ij} = -\eta \frac{\partial E_p}{\partial w_{ij}} = \eta (t_i^p - o_i^p) f'(u_i)x_{ij}$$

where $f'(u_i) = \frac{df}{du_i}$ is the derivative of $f$ with respect to $u_i$. If $f$ is the sigmoid function, $f'(u_i) = af(u_i)(1 - f(u_i))$. 
We can update weights after processing each pattern, using rule:

\[ \Delta w_{ij} = \eta \left( t^p_i - o^p_i \right) f'(u_i) \times_{ij} \]

\[ \Delta w_{ij} = \eta \delta^p_i \times_{ij} \]

- This is known as the generalized delta rule.
- We need to use the derivative of the activation function \( f \).
  So, \( f \) must be differentiable!
- Sigmoid has a derivative which is easy to calculate.
Using Gradient Descent to Minimize the Error

We can update weights after processing each pattern, using rule:

\[ \Delta w_{ij} = \eta (t^p_i - o^p_i) f'(u_i) x_{ij} \]
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- This is known as the generalized delta rule.
- We need to use the derivative of the activation function \( f \). So, \( f \) must be differentiable!
- Sigmoid has a derivative which is easy to calculate.
Updating Output vs Hidden Neurons

We can update output neurons using the generalize delta rule:

$$\Delta w_{ij} = \eta \, \delta_i^p \, x_{ij}$$

$$\delta_i^p = (t_i^p - o_i^p) f'(u_i)$$

This $\delta_i^p$ is only good for the output neurons, it relies on target outputs. But we don’t have target output for the hidden nodes!

$$\Delta w_{ki} = \eta \, \delta_k^p \, x_{ik}$$

$$\delta_k^p = \sum_{j \in I_k} \delta_j^p \, w_{kj}$$

This rule propagates error back from output nodes to hidden nodes. If effect, it blames hidden nodes according to how much influence they had. So, now we have rules for updating both output and hidden neurons!
1. Present the pattern at the input layer.
1. Present the pattern at the input layer.
2. Propagate forward activations
1. Present the pattern at the input layer.
2. Propagate forward activations step
1. Present the pattern at the input layer.
2. Propagate forward activations step by step.
1. Present the pattern at the input layer.
2. Propagate forward activations step by step.
Backpropagation

1. Present the pattern at the input layer.
2. Propagate forward activations step by step.
3. Calculate error from both output neurons.
1. Present the pattern at the input layer.
2. Propagate forward activations step by step.
3. Calculate error from both output neurons.
4. Propagate backward error.
1. Present the pattern at the input layer.
2. Propagate forward activations step by step.
3. Calculate error from both output neurons.
4. Propagate backward error.
1. Present the pattern at the input layer.
2. Propagate forward activations step by step.
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4. Propagate backward error.
1. Present the pattern at the input layer.
2. Propagate forward activations step by step.
3. Calculate error from both output neurons.
4. Propagate backward error.
5. Calculate $\frac{\partial E}{\partial w_{ij}}$; repeat for all patterns and sum up.
Online Backpropagation

1: Initialize all weights to small random values.
2: repeat
3: for each training example do
4: Forward propagate the input features of the example to determine the MLP’s outputs.
5: Back propagate error to generate $\Delta w_{ij}$ for all weights $w_{ij}$.
6: Update the weights using $\Delta w_{ij}$.
7: end for
8: until stopping criteria reached.
• We learnt what a multilayer perceptron is.
• We have some intuition about using gradient descent on an error function.
• We know a learning rule for updating weights in order to minimize the error: \( \Delta w_{ij} = -\eta \frac{\partial E}{\partial w_{ij}} \)
• If we use the squared error, we get the generalized delta rule: \( \Delta w_{ij} = \eta \delta_{i}^{p} x_{ij} \).
• We know how to calculate \( \delta_{i}^{p} \) for output and hidden layers.
• We can use this rule to learn an MLP’s weights using the backpropagation algorithm.