Notes to Answers assignment 2: Variants of the BCM-rule

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In this assignment we consider the BCM plasticity rule for a single neuron receiving just two inputs.

Given the two-dimensional input vector $\boldsymbol{x}$, the neuron's activity is $y = \boldsymbol{w}.\boldsymbol{x}$, where $\boldsymbol{w}$ are the synaptic weights, written as a vector. The activities as well as weights are allowed to be negative.

The weights are plastic according to the BCM -rule

$$\frac{d\boldsymbol{w}}{dt} = \eta \boldsymbol{x} y(y - \theta)$$

The threshold θ tracks the recent value of the activity y^2, and obeys

$$\tau_\theta \frac{d\theta}{dt} = -\theta + y^2$$

The input alternates between two values $\boldsymbol{x}^a = (\cos\phi_a, \sin\phi_a)$ and $\boldsymbol{x}^b = (\cos\phi_b, \sin\phi_b)$, choose $\phi_a = 0$, $\phi_b = \pi/3$.

Simulations of the system contain the following steps: select the input, calculate the neural activity, update the weights, and update the threshold.

Parameters: weight update $\eta = 0.01$, $\tau_\theta = 10$ (timesteps), at least 1000 total simulation steps.

Questions

Note each question can maximally achieve 25 pts.

Question 1 Simulate the system and plot the resulting activity, threshold and weights versus time. Comment on your findings.

A: The system converges to one of two stable fixed points (possibly after some damped oscillation). You should find that depending on initial conditions, w converges to either of two configurations: w=(0,2.4) or w=(2.1,-1.2).

In either case one input will lead to no activity, the other to high activity (i.e. the solution is highly selective). See next question.

Note, if you start with both w's negative, you might need to simulate very long (~50000 steps). Note, there are no units (other than time-step).

Question 2 Show mathematically, that unless $\boldsymbol{x}^a \propto \boldsymbol{x}^b$, in the steady state the weights are such that $\boldsymbol{x}^a.\boldsymbol{w}$ equals zero or θ, and $\boldsymbol{x}^b.\boldsymbol{w}$ equals zero or θ (you can assume a fixed θ for this). Compare to your simulation results.

A: The equations for stability are $\Delta w_1^a + \Delta w_1^b = 0$, where Δw_1^a is the change in weight 1 in response to pattern a, and same equation for Δw_2. Filling in the BCM rule: $x_1^a y^a(y^a - \theta) + x_1^b y^b(y^b - \theta) = 0$. Note that when we don't have that $\boldsymbol{x}^a \propto \boldsymbol{x}^b$, we have the non-degenerate vector equation, $c\boldsymbol{x}^a + d\boldsymbol{x}^b = \boldsymbol{0}$, with $c = y^a(y^a - \theta)$, this can only hold if $c = d = 0$. So $y^a(y^a - \theta) = y^b(y^b - \theta) = 0$.

Note that this analysis does tell us whether the fixed points are stable or not. In comparing with the simulations, you will find indeed these fixed points, but in addition you should remark that apparently out of the four possible fixed points, only the two asymmetric fixed points are stable.

Note, it is not necessary that $\Delta w_1^a = \Delta w_1^b = 0$.

Question 3 Explore what happens in the simulations if you make τ_θ larger and comment on your findings.

A: Simulation shows that there is a critical, maximal value for τ_θ, of about 150 timesteps, below that value you will find damped oscillations, but above that value, oscillations never stop. For even slower τ_θ, one gets spikes of activity: short bouts of very high activity interleaved by long periods of very low activity (of course not at all related to neural spikes!). (Note, also very quick τ_θ ($\tau_\theta \leq 1$) can lead to oscillations.)

The analysis above is not valid any more because we can't assume θ fixed. The intuition is that when θ is not updated fast enough, the plasticity and activity can 'escape'. Eventually θ will catch up because it is proportional to y^2. (For more formal analysis see Udeigwe, Munro & Ermentrout, 2017).

Question 4 As variant of the rule simulate the following rule $\frac{dw_i}{dt} = \eta x_i y(y - \theta|w_i|)$. What are the steady state weights under that rule?

A: In the case you should find that the solutions subtly change: there is one fixed point, where both input patterns lead to some activity (i.e. much less selective than before) at $w = (0.85, 1.05)$. There is another fixed point at (1.4,-0.8); this one is like the selective ones above, where one input pattern leads to zero activity.

Analytical solutions to the steady state equations are very complicated (if they exist at all), even under the assumption that the weights remain positive so that the absolute sign can be ignored. Again for slow τ_θ, instabilities emerge, but this occurs at much larger values of τ_θ than before.