Bayesian Logistic Regression

Different plausible decision boundaries

\[ p(y=1 | x, w) = \sigma(w^T x) = \frac{1}{2} \]

for different \( w \sim \text{posterior} \)

Posterior

\[ p(w | D, M) = \frac{p(D | w, M) p(w | M)}{p(D | M)} \]

Likelihood: Large product sigmoids

Predictions

\[ P(y | x, D) = \int p(y | x, w) p(w | D, M) dw \]

Test input

Marginal Likelihood

\[ \int p(D | w) p(w) dw \]
Posterior $p(w | D)$ within some model with fixed basis functions prior on weights

Contour of $p(y=1 | x, w, \text{basis fn's})$

Prediction

Marginal Likelihood $p(D)$
Importance Sampling (special case, using prior)

\[ P(y \mid x, D) = \frac{\int p(y \mid x, w) P(D \mid w) p(w) \, dw}{P(D)} \approx \int p(D \mid w) p(w) \, dw \]

\[ \approx \frac{1}{S} \sum_{s=1}^{S} p(y \mid x, w^{(s)}) P(D \mid w^{(s)}) \]

\[ \frac{1}{S} \sum_{s=1}^{S} p(D \mid w^{(s)}) \]

\[ w^{(s)} \sim \text{prior } p(w) \]

Laplace Approximation

\[ p(w \mid D) \approx \mathcal{N}(w; \hat{w}, H^{-1}) \]

\[ \arg\min_{w} E(w) \]

\[ H_{ij} = \frac{\partial^2 E}{\partial w_i \partial w_j} \]

"Energy", \( E = -\log p(w, D) + \text{const wrt } w \)

If posterior is Gaussian, is correct.
Otherwise an approximation ("wrong"!)

Predictions for logistic regression

\[ p(y=1 \mid x, D) \propto \int p(y=1 \mid x, w) N(w; w^*, H^{-1}) dw \]

\[ = \int \sigma(w^T x) N(w; w^*, H^{-1}) dw \]

\[ = \mathbb{E}_{N(w; w^*, H^{-1})} \left[ \sigma(w^T x) \right] \]

(Could do Monte Carlo)

Average under an "activation" \( w^T x = a \)

\[ = \mathbb{E}_p(a) \left[ \sigma(a) \right] \]

\[ = \mathbb{E}_{N(a; w^*^T x, x^T H^{-1} x)} \left[ \sigma(a) \right] \]

Could solve numerically

\[ \propto \sigma \left( \kappa w^T x \right) \]

Murphy § 8.4.4.2

\[ \kappa = \frac{1}{\sqrt{1 + \frac{\pi}{8} x^T H^{-1} x}} \]
\[ f = w^T x \]
\[ f = \sigma(w^T x) \]
Approx. Normalizer $p(D)$

\[ p(w | D) = \frac{p(w, D)}{p(D)} \propto N(w; w^*, H^{-1}) \]

\[
\frac{1}{p(D)} = \frac{1}{|H|^{D/2}} \exp\left(-\frac{1}{2} (w - w^*)^T H (w - w^*)\right)
\]

Evaluate approx. at $w = w^*$:

\[
\frac{p(w^* | D)}{p(D)} \propto \frac{1}{|H|^{D/2}}
\]

$\uparrow$ Training set

\[ P(D) \approx \frac{p(w^* | D)}{p(w | D)} \propto \frac{p(w^* | D)}{|H|^{D/2}} \]

True $p(w | D)$

$w^*$

$\rightarrow w$ multimodal

$P(D)$ approx with Laplace

A) Too Big, B) Too Small, C) Correct, Z) ????
Regression

Sketch logistic posterior for 1 data point

\[ p(w) = N(w; 0, 1) \]
\[ x = -20 \]
\[ y = +1 \]

\[ p(w|D) \propto N(w; 0, 1) \sigma(10 - 20w) \]
\[ \uparrow \text{Known bias} = 10 \]

likelihood