1. Consider the following Prolog program and query.

\[
\begin{align*}
n(e,\_).
n(f(X),g(Y)) & :- n(X,Y).
n(g(X),f(Y)) & :- n(X,Y).
\end{align*}
\]

? - n(X, f(g(e))), n(X, f(f(e))).

(a) Write out this program and query in the notation of first-order logic, making all quantification explicit.
(b) Draw the full search tree generated by the program and query.

2. Consider the three substitutions displayed below.

\[
\begin{align*}
\{ X = f(Y) \} \\
\{ X = f(g(Z), w) \} \\
\{ X = f(g(Z), g(Z)) \}
\end{align*}
\]

For each pair of substitutions from the above set, say whether one of the substitutions is more general than the other, and justify your answer.

3. For each of the following pairs of terms, find two different unifiers \( \theta_1 \) and \( \theta_2 \) such that \( \theta_1 \) is a most-general unifier, and such that \( \theta_2 \) is not most general.

(a) \( f(X, h(Z)) \) and \( f(g(W), U) \)
(b) \( f(X, h(Z)) \) and \( f(g(W), h(W)) \)

In what sense are the most general unifiers unique?

4. This question concerns the resolution proof method, introduced by J. A. Robinson; and SLD resolution introduced by R. Kowalski.

In predicate logic a literal is a formula that is either an atomic formula or the negation of an atomic formula. That is, a literal is either of the form \( A \) or of the form \( \neg A \), where \( A \) is an atomic formula. A literal of the form \( A \) is said to be a positive literal; one of the form \( \neg A \) is said to be negative.

A clause is a formula that is a finite disjunction

\[
L_1 \lor \cdots \lor L_k
\]

of literals \( L_1, \ldots, L_k \), where \( k \geq 0 \). We write the \( k = 0 \) clause as \( \bot \); it can be thought of as representing the truth value false.
The *resolution proof rule* combines two clauses as follows:

\[
\begin{align*}
L_1 \lor \cdots \lor L_k & \quad L'_1 \lor \cdots \lor L'_n \\
\rightarrow \quad (L_1 \lor \cdots \lor L_{l-1} \lor L_{l+1} \lor \cdots \lor L_k \lor L'_1 \lor \cdots \lor L'_{m-1} \lor L'_{m+1} \lor \cdots \lor L'_n) \theta
\end{align*}
\]

where \(1 \leq l \leq k\), \(1 \leq m \leq n\), \(L_i\) is an atomic formula \(A\), \(L'_m\) is a negated atomic formula \(\neg B\), and \(\theta\) is the most general unifier of \(A\) and \(B\).

A *resolution refutation* for a finite set of clauses \(C_1, \ldots, C_N\), where \(N \geq 1\), is a tree of applications of the proof rule, for which the leaves are (variable renamings of) clauses in \(C_1, \ldots, C_N\), and the root is the empty clause \(\bot\).

Resolution refutations are sound and complete in the sense that the following statements are equivalent.

(i) \(C_1, \ldots, C_N\) has a resolution refutation.

(ii) \(\forall \text{Vars}(C_1). C_1 \land \cdots \land \forall \text{Vars}(C_N). C_N \models \text{false}\).

Statement (ii) is equivalently expressed as saying that the formula below is unsatisfiable (i.e., it has no models).

\[\forall \text{Vars}(C_1). C_1 \land \cdots \land \forall \text{Vars}(C_N). C_N\]

SLD resolution is a special case of resolution. A *definite clause* is a clause that has exactly one positive literal. A *goal clause* is a clause that has no positive literals. The notion of an *SLD-resolution refutation* is defined for a set of clauses:

\[G, D_1, \ldots, D_N\]

where \(G\) is a single goal clause and \(D_1, \ldots, D_N\) are definite clauses.

**Definition.** An *SLD-resolution refutation*, for clauses \(G, D_1, \ldots, D_N\) as above, is a resolution refutation that satisfies the additional condition that, in each application of the resolution rule (1), the clause \(L'_1 \lor \cdots \lor L'_n\) is required to be goal. (This implies that \(L_1 \lor \cdots \lor L_k\) must always be one of the definite clauses in \(D_1, \ldots, D_N\), and thus the conclusion clause \((L_1 \lor \cdots \lor L_{l-1} \lor L_{l+1} \lor \cdots \lor L_k \lor L'_1 \lor \cdots \lor L'_{m-1} \lor L'_{m+1} \lor \cdots \lor L'_n) \theta\) is also goal.)

Work out the one-to-one correspondence between SLD-resolution refutations, as defined above, and derivations in the inference system defined in Theory Lecture 5. Your answer should consider, in turn: how \(D_1, \ldots, D_N\) corresponds to a Prolog program; how \(G\) corresponds to a Prolog goal; how the unsatisfiability of the conjunction of \(G, D_1, \ldots, D_N\) (as in (ii) above) corresponds to the existentially quantified Prolog goal being a logical consequence of the universally quantified Prolog program; and, finally, the correspondence between refutations and derivations.

5. (a)* Prove technical lemmas 1 and 2, from Theory Lecture 5.

(b)** Formulate and prove analogues of technical lemmas 1 and 2 for general Robinson-style resolution refutations from question 4 above.