Learning from Data, Tutorial Sheet for Week 2: Answers

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1. Calculate the length of the vector $(1, -1, 2)$.

Vector length - Euclidian Norm

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}.\mathbf{v}} \text{ hence } \|\mathbf{v}\| = \sqrt{6}$$

2. Show that $\sum_i w_i x_i = |\boldsymbol{w}||\boldsymbol{x}| \cos\theta$, where θ is the angle between the two vectors $\boldsymbol{w}$ and $\boldsymbol{x}$.

Angle between vectors

Vectors $\mathbf{x}$, $\mathbf{w}$ and $(\mathbf{x} - \mathbf{w})$ lie along the three sides of a triangle. Hence by the cosine rule:

$$|\mathbf{x} - \mathbf{w}|^2 = |\mathbf{x}|^2 + |\mathbf{w}|^2 - 2|\mathbf{x}||\mathbf{w}| \cos(\theta)$$

The result follows straightforwardly.

3. Let A and $\boldsymbol{v}$ be defined as

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

Calculate $A\boldsymbol{v}$. Is $\boldsymbol{v}$ an eigenvector of A, and if so what is the corresponding eigenvalue?

Matrix operations, eigenvalues and eigenvectors.

If $A\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{v}$ is an eigenvector of A and λ is the associated *eigenvalue*. Solution $\mathbf{v}$ is an *eigenvector* of A with $\lambda = -6$.

4. Partial derivatives. Find the partial derivatives of the function $f(x, y, z) = (x + 2y)^2 \sin(xy)$.

Partial Derivatives

$$\frac{\partial f}{\partial x} = y(x + 2y)^2 \cos(xy) + 2(x + 2y) \sin(xy)$$
$$\frac{\partial f}{\partial y} = x(x + 2y)^2 \cos(xy) + 4(x + 2y) \sin(xy)$$
$$\frac{\partial f}{\partial z} = 0$$

5. Probability. Lois knows that on average radio station RANDOM-FM plays 1 out of 4 of her requests. If she makes 3 requests, what is the probability that at least one request is played?

Probability.

Random variable X denotes the "number of requests played". Then

$$P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) = 1 - P(X = 0)$$
$$P(X = 0) = (0.75)^3$$
$$P(X \geq 1) = 0.5781$$

6. Let X be distributed according to a uniform distribution, i.e. $f(x) = 1$ for $0 \leq x \leq 1$, and 0 otherwise. Show that X has mean $\frac{1}{2}$ and variance $\frac{1}{12}$.

Distribution Functions

$$EX = \int_{-\infty}^{\infty} dx\ x f(x)$$

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X) = 1/2,\ Var(X) = 1/12$$

7. Find the (unconstrained) minimum of the function

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \boldsymbol{b}^T \boldsymbol{x}$$

where A is a positive definite symmetric matrix.

Find the minimum of $f(\boldsymbol{x})$ along the line $\boldsymbol{x} = \boldsymbol{a} + t\boldsymbol{v}$ where t is a real parameter.

Calculate explicitly the unconstrained minimum in the case that

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Optimisation

If A is a positive symmetric matrix then the inverse A^{-1} exists.

$$(AB)^T = B^T A^T$$

The gradient of a scalar function Φ of $\mathbf{x} = (x_1, \ldots, x_n)^T$ is defined as

$$\nabla\Phi(\mathbf{x}) = \frac{\partial\Phi}{\partial x_1}\mathbf{x}_1 + \ldots + \frac{\partial\Phi}{\partial x_n}\mathbf{x}_n$$

where $\mathbf{x}_i$ is the unit vector with a one in row i.

Hence we have the solutions

a) Unconstrained minimum:

$$\nabla f(\mathbf{x}) = 0$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

b) Minimum of f along line. Substitute the line equation $\mathbf{x} = \mathbf{a} + t\mathbf{v}$ in and set the derivative w.r.t t to zero.

$$t = \frac{\mathbf{b}^T\mathbf{v} - \mathbf{a}^T A\mathbf{v}}{\mathbf{v}^T A\mathbf{v}}$$

Unconstrained minimum for given A and $\mathbf{b}$ is $\mathbf{x} = (7, -3)^T$

Additional question 8. Consider the integral

$$I = \int_{\infty}^{\infty} e^{-\frac{1}{2}x^2} dx$$

By considering

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} dx dy$$

and converting to polar coordinates, r, θ, show that

$$I = \sqrt{2\pi}$$

Easy!

$r^2 = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$. Jacobian is r. New limits are r from 0 to ∞ and θ from 0 to 2π. Then the integral splits into two:

$$\int_0^{2\pi} d\theta = 2\pi$$

and

$$\int_0^{\infty} dr \ r \exp^{-frac12r^2} = 1$$

Hence $I^2 = 2\pi$, $I = \sqrt{(2\pi)}$.