This week’s questions go beyond the course, and are intentionally difficult. If you can do them without looking things up, you should probably be teaching the course.

(1) In our definitions of polymorphic type inference, we used a $\forall$ sign: $\forall \alpha. \alpha \to \alpha$, for example. Can you think of a way for a type $\exists \alpha. \alpha$ to make sense? Hint: logically, an $\exists$ statement is saying not only that there exists a type, but it’s also saying that we don’t know what it is (yet). In order to demonstrate the truth of $\exists x. \phi(x)$, we have to instantiate $x$ with some concrete value. Does this remind you of anything in programming languages?

(2) You’re used to the idea of polymorphic lists in Haskell or ML. Clearly (!), the polymorphic head function has type $\forall \alpha. (\text{List} \ \alpha) \to \alpha$. But what is List? It’s something that takes a type, and gives back a type. It is a function on types! Similarly, $\to$ is a function that takes two types and returns a type.

So we can imagine writing a $\lambda$-calculus extended with functions on types: for example, $\text{Pair} = \Lambda \alpha. \Lambda \beta. \alpha \times \beta$ might define a Pairing constructor in terms of product types.

Do we need some notion of ‘type of types’? What would the rules look like for these? What would be base ‘types of types’ be? What’s the difference between $\Lambda \alpha$ in this ‘type constructor’, and $\forall \alpha$ in polymorphic types?

(3) This question assumes the non-examinable Lecture 16, which was not delivered this year. If you want to investigate it yourself, look up ‘Curry–Howard correspondence’, for example in its Wikipedia article, particularly the section ‘Natural deduction and lambda calculus’.

In intuitionistic logic, negation is defined as $\neg \phi \overset{\text{def}}{=} \phi \to \bot$, where $\to$ is logical implication, and $\bot$ is ‘false’. Thus, a proof of $\neg \phi$ is a procedure that from a proof of $\phi$ produces a proof of $\bot$.

What does negation look like under the Curry–Howard correspondence? That is, if $\tau$ is a type, what is the type $\neg \tau$, and what is a term witnessing $\neg \tau$?