Note 2 – doing reductions

I was asked for an example of how one might write down a reduction for examination purposes. Here’s an example from the slides written out in a way that I would give full marks to on an exam.

**Question:** Show, by reduction from Halting, that the Uniform Halting problem is undecidable.

**Answer:** It is a theorem that if $Q$ can be reduced by a many-one (or Turing) reduction to $Q'$, and $Q$ is undecidable, then $Q'$ is undecidable.

Given an instance $(M, R)$ of Halting, where $M$ is the program of a machine and $R$ its input, construct an instance of Uniform Halting as follows: let $M'$ be a machine which ignores its input, and behaves as $M$ on input $R$. It is clear that the construction of $M'$ is computable, by, e.g., starting $M'$ with code that loads $R$ into its registers and then jumps to the start of $M$.

For a reduction, we need that $(M, R) \in H \iff M' \in UH$. This is true by construction: if $(M, R)$ halts, then $M'$, which behaves as $M$ on $R$, halts whatever its input; conversely, if $M'$ halts on all (or indeed any) input, then $M$ halts on $R$.

Thus we have a reduction and have shown the result.

If you don’t like writing words, here’s a reduced verbiage version, which is ok if it’s correct, but runs the risk of getting less partial credit if it has mistakes!

**Answer:** We know if $Q \leq_m Q'$ and $Q$ undecidable, then $Q'$ undecidable. Let $(M, R)$ be a machine and input. Define $M'$ to ignore its input and run $M$ on $R$. The function $(M, R) \mapsto M'$ is clearly computable, and $M'$ is an instance of $UH$. By construction, $M'$ halts on any input iff $M$ halts on $R$. Thus $H \leq_m UH$ and we are done.