Introduction to Theoretical Computer Science

Lecture 8: Semi-decidability

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A Review of Halting

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Definition

A problem (D, Q) is *semi-decidable* if there is a TM/RM that returns "yes" for any $d \in Q$, but may return "no" or loop forever when $d \notin Q$.

Semi-decidable problems are sometimes called *recognisable*.

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Theorem

Any problem that is both semi-decidable and co-semi-decidable is decidable. **Why?**

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$\therefore$ L cannot be semi-decidable. **Why?**

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Are there any problems that are neither semi-decidable nor co-semi-decidable?

Yes, by the counting argument from last lecture.

Enumeration

Recall

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Question: H is not decidable. But can we enumerate it?

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Interleaving

Observe that the set of valid RM descriptions is decidable: Given $n \in \mathbb{N}$, we can check whether $n = \ulcorner M \urcorner$ for some machine M.

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Interleaving

ms $:= \langle\rangle$; $i := 0$
while true **do**
  ms := ms ++ $\langle \ulcorner M_i \urcorner \rangle$
  for $\ulcorner M \urcorner \in$ ms **do**
    run M for one step and update ms
    if *M has halted* :
      output $\ulcorner M \urcorner$; *delete M from* ms
  od
  $i := i + 1$
od

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Why?

Therefore, semidecidability is the same as c.e.-ness.

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Note we must use mapping reductions, not Turing reductions. Why?
H is c.e. but its complement L is not. But H is Turing-reducible to L and L is Turing-reducible to H by flipping the answers.

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- We also have a mapping reduction from L to UH, in the next slide.

Conclusion

UH is neither semi-decidable nor co-semi-decidable.

We showed that such problems must exist earlier by counting.

L to UH

We want a transducer $f : RM \times \text{Input} \to RM$ such that $f(M, i)$ halts on all inputs iff M loops on i.
As with H to UH, we can make a machine that replaces any input with i and then runs M, but how do we make it stop?

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If M loops on i, then $f(M, i)$ will halt on all inputs n, and if M halts, then $f(M, i)$ loops on some sufficiently large n.

Next topics

Next week is a “low-pressure” week. We will only have one lecture, which slightly extends the theory presented here to present the fascinating arithmetic hierarchy. It’s not officially part of the syllabus of this course, but a related concept, the polynomial hierarchy is, and so I highly recommend learning it anyway.
There is no lecture on Thursday.