Introduction to Theoretical Computer Science

Lecture 14: λ-Calculus

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Introduction

While Turing was thinking about machines, Alonzo Church was computing with a programming language – a precursor of Haskell – called *λ-calculus*.

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Comparing models

Church and Turing famously proved that Turing Machines and λ-calculus are equivalent in computational power.

However, λ-calculus is different from other models in that it is *higher-order*: This means that computations (λ-terms) may take other computations as input. For TMs and RMs, we must work with encodings to achieve this.

Syntax

λ-calculus computations are expressed as *λ-terms*:

$$
\begin{array}{rcll}
t & ::= & x & \textit{(variables)} \\
  & | & t_1\ t_2 & \textit{(application)} \\
  & | & \lambda x.\ t & \textit{(}\lambda\textit{-abstraction)}
\end{array}
$$

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λ-abstraction

A λ-term $(\lambda x.\ y)$ can be thought of as a function that, given an input bound to the variable x, returns the term y.
We will give a formal definition of this in terms of *substitution* later.

For now, we will extend λ-terms with arithmetic expressions:

$$(\lambda x.\ \lambda y.\ (x + y) \div 2)\ 10\ 20$$

but this is not fundamental to the computational model. We will remove this feature later without reducing expressivity.

Higher-order functions

Function application is left associative:

$$f\ a\ b\ c \quad = \quad ((f\ a)\ b)\ c$$

λ-abstraction extends as far as possible:

$$\lambda a.\ f\ a\ b \quad = \quad \lambda a.\ (f\ a\ b)$$

All functions are unary, like Haskell. Multiple argument functions are modelled with nested λ-abstractions:

$$\lambda x.\ \lambda y.\ f\ y\ x$$

λ-calculus is *higher-order*, in that functions may be arguments to functions themselves:

$$\lambda f.\ \lambda g.\ \lambda x.\ f\ (g\ x)$$

α-equivalence

What is the difference between these two programs?

$$(\lambda x.\, \lambda x.\, x + x) \qquad (\lambda a.\, \lambda y.\, y + y)$$

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They are semantically identical, but differ in the choice of bound variable names. Such expressions are called *α-equivalent*.

We write $e_1 \equiv_\alpha e_2$ if e_1 is α-equivalent to e_2. The relation $\equiv_\alpha$ is an *equivalence relation*.
The process of consistently renaming variables that preserves α-equivalence is called *α-renaming* or *α-conversion*.

Substitution

A variable x is *free* in a term e if x occurs in e but is not *bound* (by a λ-abstraction) in e.

Example (Free Variables)

The variable x is free in $\lambda y.\, x + y$, but not in $\lambda x.\, \lambda y.\, x + y$.

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A *substitution*, written $e[{}^{t}/_{x}]$, is the replacement of all free occurrences of x in e with the term t.

Example (Substitution on Arithmetic Expressions)

$(5 \times x + 7)\left[{}^{y \times 4}/_{x}\right]$ is the same as $(5 \times (y \times 4) + 7)$.

Problems with substitution

Consider these two α-equivalent expressions.

$$(\lambda y.\, y \times x + 7)\ 5$$

and

$$(\lambda z.\, z \times x + 7)\ 5$$

What happens if you naïvely apply the substitution $\left[{}^{y \times 3}/_{x}\right]$ to both expressions?

Problems with substitution

Consider these two α-equivalent expressions.

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What happens if you naïvely apply the substitution $[{}^{y \times 3}/_{x}]$ to both expressions? You get two non-α-equivalent expressions!

$$(\lambda y.\, y \times (y \times 3) + 7)\ 5$$

and

$$(\lambda z.\, z \times (y \times 3) + 7)\ 5$$

This problem is called *capture*.

Variable Capture

Capture can occur for a substitution $e\left[{}^{t}/_{x}\right]$ whenever there is a bound variable in the term e with the same name as a free variable occuring in t.

Fortunately

It is always possible to avoid capture. Just α-rename the offending bound variable to an unused name.

β-reduction

The rule to evaluate function applications is called *β-reduction*:

$$(\lambda x.\ t)\ u \quad \mapsto_\beta \quad t\left[{}^{u}/_{x}\right]$$

β-reduction

β-reduction is a *congruence*:

$$\frac{}{(\lambda x.\ t)\ u \mapsto_\beta t\left[{}^{u}/_{x}\right]}$$

$$\frac{t \mapsto_\beta t'}{s\ t \mapsto_\beta s\ t'} \qquad \frac{s \mapsto_\beta s'}{s\ t \mapsto_\beta s'\ t} \qquad \frac{t \mapsto_\beta t'}{\lambda x.\ t \mapsto_\beta \lambda x.\ t'}$$

This means we can pick any reducible subexpression (called a *redex*) and perform β-reduction.

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Example:

$$(\lambda x.\ \lambda y.\ f\ (y\ x))\ 5\ (\lambda x.\ x)$$

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Example:

$$\begin{aligned} (\lambda x.\ \lambda y.\ f\ (y\ x))\ 5\ (\lambda x.\ x) &\mapsto_\beta (\lambda y.\ f\ (y\ 5))\ (\lambda x.\ x) \\ &\mapsto_\beta f\ ((\lambda x.\ x)\ 5) \end{aligned}$$

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Confluence

There are often many different ways to reduce the same expression:

$$(\lambda a.\ a)\ ((\lambda y.\ f\ y)\ 5)$$

Call-by-name

$$(\lambda y.\ f\ y)\ 5$$

Evaluate function args
late
(after application)

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$$\begin{tikzcd} & (\lambda a.\, a)\ ((\lambda y.\, f\ y)\ 5) \arrow[dl, "\text{Call-by-name}"'] \arrow[dr, "\text{Call-by-value}"] & \\ (\lambda y.\, f\ y)\ 5 & & (\lambda a.\, a)\ (f\ 5) \end{tikzcd}$$

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(\lambda y.\ f\ y)\ 5 \arrow[dr] & & (\lambda a.\ a)\ (f\ 5) \arrow[dl] \\
 & f\ 5 &
\end{tikzcd}
$$

Evaluate function args **late** (after application)

Evaluate function args **early** (before application)

The Church-Rosser Theorem

If a term t β-reduces to two terms a and b, then there is a common term t' to which both a and b are β-reducible.

Equivalence

Confluence means we can define another notion of *equivalence*, which equates more than α-equivalence. Two terms are $\alpha\beta$-*equivalent*, written $s \equiv_{\alpha\beta} t$ if they β-reduce to α-equivalent terms.

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η

There is also another equation that cannot be proven from β-equivalence alone, called η-reduction:

$$(\lambda x.\ f\ x) \mapsto_\eta f$$

Adding this reduction to the system preserves confluence (and therefore uniqueness of normal forms), so we have a notion of *$\alpha\beta\eta$-equivalence* also.

Normal Forms

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Divergence

Does every term in λ-calculus have a normal form?

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Does every term in λ-calculus have a normal form?

$$(\lambda x.\ x\ x)(\lambda x.\ x\ x)$$

Try to β-reduce this!

Uniqueness of NFs

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Divergence

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$$(\lambda x.\ x\ x)(\lambda x.\ x\ x)$$

Try to β-reduce this!

Uniqueness of NFs

Does any term in λ-calculus have more than one normal form?
No: consider Church-Rosser.

Making λ-Calculus Usable

In order to demonstrate that λ-calculus is actually a usable programming language, we will demonstrate how to encode booleans and natural numbers as λ-terms, along with their operations.

General Idea

We transform a data type into the type of its *eliminator*. In other words, we make a function that can serve the same purpose as the data type at its use sites.

Booleans

How do we use booleans?

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So, a boolean will be a function that, given two arguments, returns the first one if it is true and the second one if it is false:

$$\begin{aligned} \text{True} &\equiv \lambda a.\, \lambda b.\, a \\ \text{False} &\equiv \lambda a.\, \lambda b.\, b \end{aligned}$$

How do we write an if statement?

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How do we write an if statement?

$$\text{If} \equiv \lambda c.\, \lambda t.\, \lambda e.\, c\ t\ e$$

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Example (Test it out!)

Try β-normalising If True False True.

Natural Numbers

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So, a natural number will be a function that takes a function f and a value x, and applies the function f to x that number of times:

$$
\begin{aligned}
\text{Zero} &\equiv \lambda f.\, \lambda x.\, x \\
\text{One} &\equiv \lambda f.\, \lambda x.\, f\ x \\
\text{Two} &\equiv \lambda f.\, \lambda x.\, f\ (f\ x) \\
&\dots
\end{aligned}
$$

How do we write Suc?

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How do we write Suc?

$$\text{Suc} \equiv \lambda n.\, \lambda f.\, \lambda x.\, f\ (n\ f\ x)$$

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How do we write Add?

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&\dots
\end{aligned}
$$

How do we write Suc?

$$\text{Suc} \equiv \lambda n.\, \lambda f.\, \lambda x.\, f\ (n\ f\ x)$$

How do we write Add?

$$\text{Add} \equiv \lambda m. \lambda n.\, \lambda f.\, \lambda x.\, m\ f\ (n\ f\ x)$$

Natural Numbers

Example

Try β-normalising Suc One.

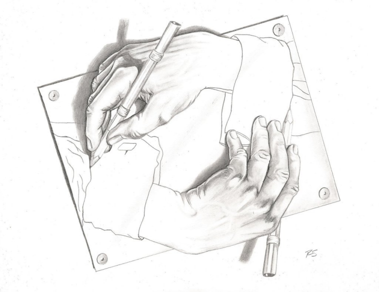
Example

Try writing a different λ-term for defining Suc.

Example

Try writing a λ-term for defining Multiply.

A Final Puzzle



Puzzle

Find a λ-term $\mathcal{Y}$ such that

$$\mathcal{Y}\, f \quad \mapsto_{\beta}^{\star} \quad f\,(\mathcal{Y}\, f)$$

Can you use this to define recursive functions? e.g. factorial?