Introduction to Theoretical Computer Science

Lecture 12: NP-Completeness

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Hardness

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A problem P is ***NP-Hard*** if, for every $A \in \mathbf{NP}$, $A \leq_P P$

- If a problem P_1 is **NP**-hard and $P_1 \leq_P P_2$ then P_2 is **NP**-Hard.
- To prove that a problem P_2 is **NP-hard**, show that there's a polynomial reduction from a known **NP-hard** P_1 to P_2.

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Do **NP**-Complete Problems Exist?

There are **many** such problems, including *HPP* and Timetabling. In fact, almost all **NP**-problems encountered in practice are either in **P**, or **NP**-complete.

Computers and Intractability - A guide to theory of NP-completeness, M.R. Garey and D.S. Johnson, Freeman 1979 lists a whole bunch.

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The Proof

- **The** *SAT* **problem is in NP**: Nondeterministically guess an assignment and check it in polynomial time.
- **The** *SAT* **problem is NP-Hard**: Shown by reduction from any **NP** problem to *SAT*.

The Reduction

Suppose $(D, Q) \in$ **NP**. We shall construct a reduction $Q \leq_P SAT$. Given an instance $d \in D$, we shall construct a formula φ_d which can be satisfied if its variables describe the successful executions of an NRM checking Q. This machine can be polynomially bounded, so the size of φ_d will be polynomial in the size of d.

The Variables

Our NRM for Q, $M = (R_0, \ldots, R_{m-1}, I_0, \ldots, I_{n-1})$ runs for s steps (i.e. $p(|d|)$ where d is our input and p is our polynomial bound).

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Why $2s$?

How big can the registers get? Running s steps of $\mathrm{ADD}(0,0)$ will make R_0 double s times, if it starts at $2^{|d|}$ then we need $2^{|d|+s}$ capacity. Then w.l.o.g. 2^{2s} i.e. $2s$ bits is enough.

The Formula

$$C_{00} \wedge \rho_{\text{init}} \wedge \chi_{\text{one}} \wedge \bigwedge_{t} \chi_t \wedge \alpha$$

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Some formulae are easy:

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$$\chi_{\text{one}} \equiv \bigwedge_t \bigvee_j \left(C_{tj} \wedge \bigwedge_{j' \neq j} \neg C_{tj'} \right)$$

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$\varphi_t \equiv \bigvee_j (C_{tj} \wedge \nu_{tj})$, where ν_{tj} is:

- $C_{t+1,j+1}$ if I_j is INC, ADD, or SUB.
- $C_{t+1,j+1} \vee C_{t+1,j'}$ if I_j is MAYBE(j')
- $\left(\left(\bigvee_k R_{tik}\right) \wedge C_{t+1,j+1}\right) \wedge \left(\left(\bigwedge_k \neg R_{tik}\right) \wedge C_{t+1,j'}\right)$ if I_j is DECJZ(i, j')

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Despite this being very tedious, these formulae are polynomial ($\mathcal{O}(s^4)$) !

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- φ is in *k-CNF* if each clause $\bigvee_j P_{ij}$ has at most k literals.

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Note: Cook-Levin for TMs is already for 3SAT.

Clique

The *CLIQUE* problem

Given a graph $G = (V, E)$ and a number k, a k-*clique* is a k-sized subset C of V , such that every vertex in C has an edge to every other. (C forms a complete subgraph.) Decide whether G has a k-clique.

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The Graph

Each x_{ij} is a vertex. Connect x_{ij} to $x_{i'j'}$ iff: $i \neq i'$ and $x_{i'j'}$ is not the negation of x_{ij}.

i.e. we connect literals in different clauses so long as they are not inconsistent.

Why does this work?

Since the vertices in one clause are disconnected, finding a k-clique amounts to finding one literal for each clause, such that they are all consistent — and so represent a satisfying assignment. Conversely, any satisfying assignment generates a k-clique.

P vs. NP

As previously mentioned, we **don't know** if **P** and **NP** are really distinct classes.

Find a polynomial time algorithm for any **NP**-hard problem and you can win yourself one million US dollars from the Clay Institute. (Also hire bodyguards because most web/banking security depends on such problems being hard)

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Example (**NP**-Intermediacy)

A problem is **NP**-Intermediate if it is in **NP** but not in **P** nor **NP**-complete.

If **P** $\neq$ **NP**, then graph isomorphism is such a problem (and there aren't many others).

NP in Practice

As far as we know, **NP** problems are just hard: need exponential search, so $\mathcal{O}(p(n) \cdot 2n)$. So how do we solve them in practice?

- Randomised algorithms are often useful. Allow algorithms to toss a coin. Surprisingly one can get randomised algorithms that solve e.g. *3SAT* in time $\mathcal{O}(p(n) \cdot \frac{4}{3}^n)$.

  (Why is this useful? $2^{100} \approx 10^{31}$, while $1.33^{100} \approx 10^{12}$)

  Catch: (really) small probability of error!
- In many special classes (e.g. sparse graphs, or almost-complete graphs), heuristics lead to fast results. See `http://satcompetition.org/` for the state of the art.

Next time...

We'll be looking at the boundaries of the class **NP**, and what lays beyond. Specifically, the classes of **coNP** and **PSPACE**, as well as the *polynomial hierarchy*, analogous to the arithmetic hierarchy we've already seen, but contained entirely within **PSPACE** decidable problems.

If time, I might mention the *sublinear* classes of **L** and **NL** as well, but these are not examinable.