Information Theory

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Week 3
Symbol codes

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(Binary) Symbol Codes

For strings of symbols from alphabet e.g.,
$x_i \in \mathcal{A}_X = \{A, C, G, T\}$

Binary codeword assigned to each symbol

A	0
C	10
G	111
T	110

$CGTAGATTACAGG$

$\downarrow$

10111110011101101100100111111

Codewords are concatentated without punctuation

Uniquely decodable

We'd like to make all codewords short

But some codes are not **uniquely decodable**

A	0
C	1
G	111
T	110

$CGTAGATTACAGG$

$\downarrow$

11111001110110110010111111

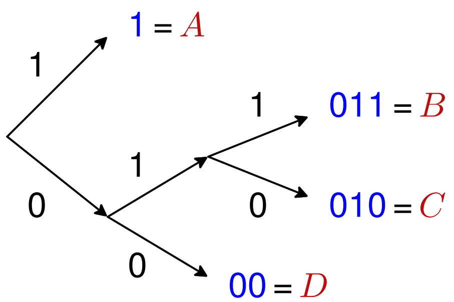
$\downarrow$

$CGTAGATTACAGG$
$CCCCCCAACCCACCACCAACACCCCCC$
$CCGCAACCCATCCAACAGCCC$
$GGAAGATTACAGG$
???

Instantaneous/Prefix Codes

Attach symbols to leaves of a binary tree

Codeword gives path to get to leaf



"Prefix code" because no codeword is a prefix of another

Decoding: follow tree while reading stream until hit leaf

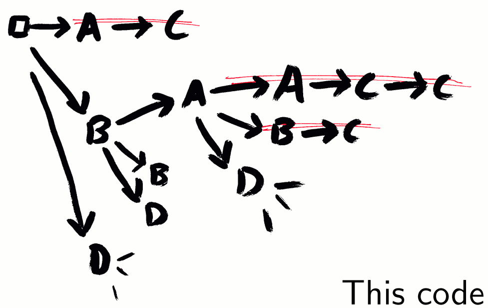
Symbol is *instantly* identified. Return to root of tree.

Non-instantaneous Codes

The last code was **instantaneously decodable**:
We knew as soon as we'd finished receiving a symbol

101100000101100

$\downarrow$



A	1
B	10
C	000
D	100

This code *is* uniquely decodable,
but not instantaneous or pleasant!

Expected length/symbol, $\bar{L}$

Code lengths: $\{\ell_i\} = \{\ell_1, \ell_2, \ldots, \ell_I\}$

$$\text{Average, } \bar{L} = \sum_i p_i\, \ell_i$$

Compare to Entropy:

$$H(X) = \sum_i p_i \log \frac{1}{p_i}$$

If $\ell_i = \log \frac{1}{p_i}$ **or** $p_i = 2^{-\ell_i}$ **we compress to the entropy**

An optimal symbol code

An example code with:

$$\bar{L} = \sum_i p_i\, \ell_i = H(X) = \sum_i p_i \log \frac{1}{p_i}$$

x	$p(x)$	codeword
A	$1/2$	0
B	$1/4$	10
C	$1/8$	110
D	$1/8$	111

Limit on code lengths

Imagine coding under an implicit distribution:

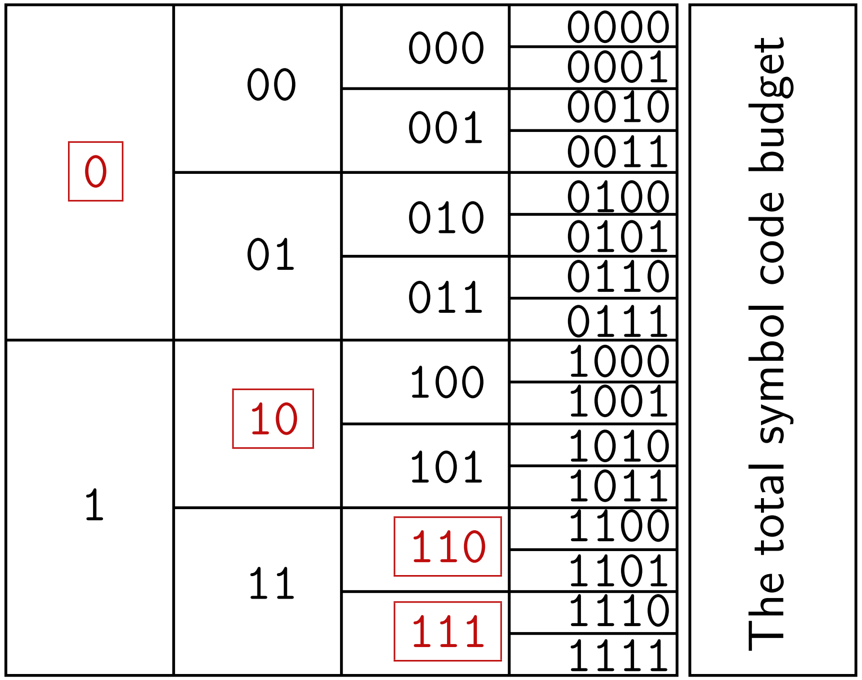
$$q_i = \frac{1}{Z} 2^{-\ell_i}, \quad Z = \sum_i 2^{-\ell_i}.$$

$$H = \sum_i q_i \log \frac{1}{q_i} = \sum_i q_i\,(\ell_i + \log Z) = \bar{L} + \log Z$$

$$\Rightarrow \log Z \le 0, \quad Z \le 1$$

Kraft–McMillan Inequality $\boxed{\sum_i 2^{-\ell_i} \le 1}$ (if uniquely-decodable)

Proof without invoking entropy bound: p95 of MacKay, or p116 Cover & Thomas 2nd Ed.



Kraft Inequality

If height of budget is 1, codeword has height $= 2^{-\ell_i}$

Pick codes of required lengths in order from shortest–largest

Choose heighest codeword of required length beneath previously-chosen code (There won't be a gap because of sorting)

Can always pick codewords if total height, $\sum_i 2^{-\ell_i} \leq 1$

Kraft–McMillan Inequality $\boxed{\sum_i 2^{-\ell_i} \leq 1}$ (instantaneous code possible)

Corollary: there's probably no point using a non-instantaneous code.
Can always make **complete code** $\sum_i 2^{-\ell_i} = 1$: slide last codeword left.

Performance of symbol codes

Simple idea: set $\ell_i = \left\lceil \log \frac{1}{p_i} \right\rceil$

These codelengths satisfy the Kraft inequality:

$$\sum_i 2^{-\ell_i} = \sum_i 2^{-\lceil \log 1/p_i \rceil} \leq \sum_i p_i = 1$$

Expected length, $\bar{L}$:

$$\bar{L} = \sum_i p_i \ell_i = \sum_i p_i \lceil \log 1/p_i \rceil < \sum_i p_i \left(\log 1/p_i + 1\right)$$

$$\bar{L} < H(\mathbf{p}) + 1$$

Symbol codes can compress to within 1 bit/symbol of the entropy.

Summary of Lecture 5

Symbol codes assign each symbol in an alphabet a codeword.
(We only considered binary symbol codes, which have binary codewords.)
Messages are sent by concatenating codewords with no punctuation.

Uniquely decodable: the original message is unambiguous

Instantaneously decodable: the original symbol can always be determined as soon as the last bit of its codeword is received.

Codeword lengths must satisfy $\sum_i 2^{-\ell_i} \leq 1$ for unique decodability

Instantaneous prefix codes can always be found (if $\sum_i 2^{-\ell_i} \leq 1$)

Complete codes have $\sum_i 2^{-\ell_i} = 1$, as realized by prefix codes made from binary trees with a codeword at every leaf.

If (big if) symbols are drawn i.i.d. with probabilities $\{p_i\}$, *and* $\ell_i = \log \frac{1}{p_i}$, then a prefix code exists that offers optimal compression.

Next lecture: how to form the best symbol code when $\{\log \frac{1}{p_i}\}$ are not integers.

Optimal symbol codes

Encode independent symbols with known probabilities:

E.g., $\mathcal{A}_X = \{A, B, C, D, E\}$
$\mathcal{P}_X = \{0.3, 0.25, 0.2, 0.15, 0.1\}$

We can do better than $\ell_i = \left\lceil \log \frac{1}{p_i} \right\rceil$

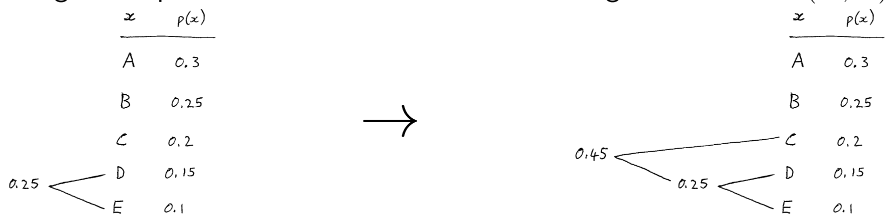
The *Huffman algorithm* gives an optimal symbol code.

Proof: MacKay Exercise 5.16 (with solution).
Cover and Thomas has another version.

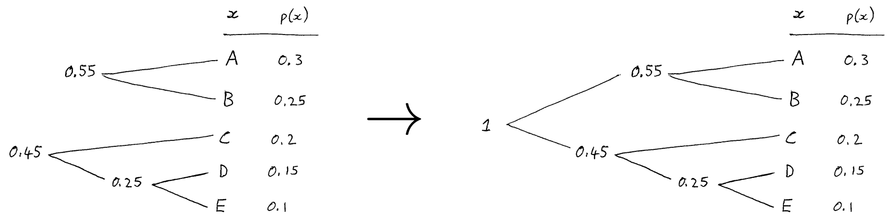
Huffman algorithm

Merge least probable

Can merge C with B or (D, E)



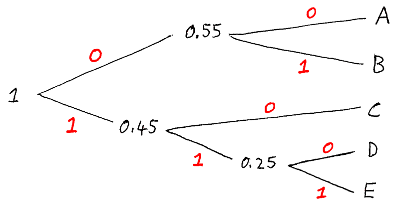
$P(D \text{ or } E) = 0.25$



Continue merging least probable, until root represents all events $P = 1$

Huffman algorithm

Given a tree, label branches with 1s and 0s to get code



x	$p(x)$	$\log_2 \frac{1}{p(x)}$	$c(x)$	$\ell(x)$
A	0.3	1.74	00	2
B	0.25	2.00	01	2
C	0.2	2.32	10	2
D	0.15	2.74	110	3
E	0.1	3.32	111	3

Code-lengths are close to the information content
(not just rounded up, some are shorter)

$H(X) \approx 2.23$ bits. Expected length $= 2.25$ bits.

Wow! Despite limitations we will discuss, Huffman codes can be very good. You'll find them inside many systems (e.g., bzip2, jpeg, mp3), although all these schemes do clever stuff to come up with a good symbol representation.

Huffman decoding

Huffman codes are easily and uniquely decodable because they are prefix codes

Reminder on decoding a prefix code stream:

— Start at root of tree
— Follow a branch after reading each bit of the stream
— Emit a symbol upon reaching a leaf of the tree
— Return to the root after emitting a symbol. . .

An input stream can only give one symbol sequence, the one that was encoded

Building prefix trees 'top-down'

Heuristic: if you're ever building a tree, consider top-down vs. bottom-up (and maybe middle-out)

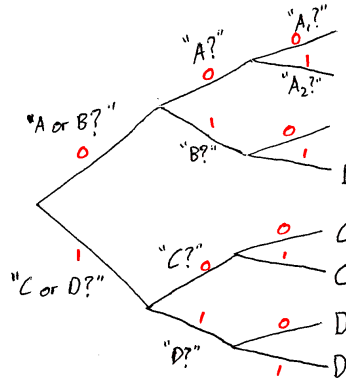
Weighing problem strategy:
Use questions with nearly uniform distribution over the answers.

How well would this work on the ensemble to the right?

x	$P(x)$
A_1	0.24
A_2	0.01
B_1	0.24
B_2	0.01
C_1	0.24
C_2	0.01
D_1	0.24
D_2	0.01

$H(X) = 2.24$ bits (just over $\log 4 = 2$). Fixed-length encoding: 3 bits

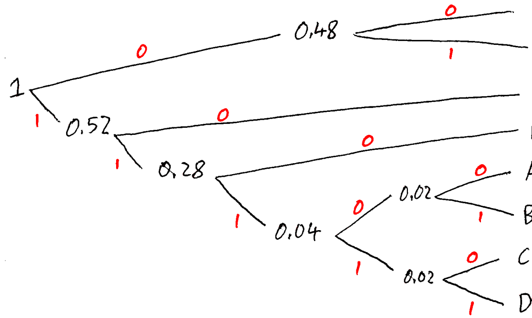
Top-down performing badly



x	$p(x)$	$c(x)$	$\ell(x)$
A_1	0.24	000	3
A_2	0.01	001	3
B_1	0.24	010	3
B_2	0.01	011	3
C_1	0.24	100	3
C_2	0.01	101	3
D_1	0.24	110	3
D_2	0.01	111	3

Probabilities for answers to first two questions is $(1/2, 1/2)$
Greedy strategy $\Rightarrow$ very uneven distribution at end

Compare to Huffman



x	$p(x)$	$c(x)$	$\ell(x)$
A_1	0.24	00	2
B_1	0.24	01	2
C_1	0.24	10	2
D_1	0.24	110	3
A_2	0.01	11100	5
B_2	0.01	11101	5
C_2	0.01	11110	5
D_2	0.01	11111	5

Expected length 2.36 bits/symbol

(Symbols reordered for display purposes only)

Relative Entropy / KL

Implicit probabilities: $q_i = 2^{-\ell_i}$

($\sum_i q_i = 1$ because Huffman codes are complete)

Extra cost for using "wrong" probability distribution:

$$\begin{aligned}\Delta L &= \sum_i p_i \ell_i - H(X)\\ &= \sum_i p_i \log 1/q_i - \sum_i p_i \log 1/p_i\\ &= \sum_i p_i \log \frac{p_i}{q_i} \;=\; D_{\mathrm{KL}}(p\,||\,q)\end{aligned}$$

$D_{\mathrm{KL}}(p\,||\,q)$ **is the** ***Relative Entropy*** **also known as the** ***Kullback-Leibler divergence*** **or** ***KL-divergence***

Gibbs' inequality

An important result:

$$D_{\mathrm{KL}}(p\,||\,q) \geq 0$$

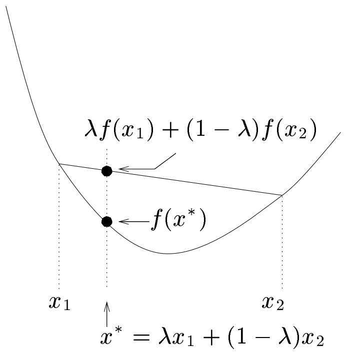
with equality only if $p = q$

"If we encode with the wrong distribution we will do worse than the fundamental limit given by the entropy"

A simple direct proof can be shown using convexity.
(Jensen's inequality)

Convexity

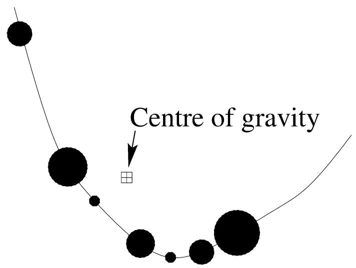
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$



Strictly convex functions:
Equality only if λ is 0 or 1, or if $x_1 = x_2$
(non-strictly convex functions contain straight line segments)

Jensen's inequality

For convex functions: $\mathbb{E}[f(x)] \geq f(\mathbb{E}[x])$



Centre of gravity at $(\mathbb{E}[x], \mathbb{E}[f(x)])$, which is above $(\mathbb{E}[x], f(\mathbb{E}[x]))$

Strictly convex functions:
Equality only if $P(x)$ puts all mass on one value

Remembering Jensen's

Which way around is the inequality?

$f(x) = x^2$ is a convex function

$\mathrm{var}[X] = \mathbb{E}[x^2] - \mathbb{E}[x]^2 \geq 0$

So we know Jensen's must be: $\mathbb{E}[f(x)] \geq f(\mathbb{E}[x])$

(Or sketch a little picture in the margin)

Convex vs. Concave

For (strictly) concave functions reverse the inequalities

For concave functions: $\mathbb{E}[f(x)] \leq f(\mathbb{E}[x])$

A (con)cave

Photo credit:
Kevin Krejci on Flickr

Jensen's: Entropy & Perplexity

Set $u(x) = \dfrac{1}{p(x)}$, $p(u(x)) = p(x)$

$\mathbb{E}[u] = \mathbb{E}[\frac{1}{p(x)}] = |\mathcal{A}|$ (Tutorial 1 question)

$H(X) = \mathbb{E}[\log u(x)] \leq \log \mathbb{E}[u]$

$H(X) \leq \log |\mathcal{A}|$

Equality, maximum Entropy, for constant $u \Rightarrow$ uniform $p(x)$

$2^{H(X)}$ = "Perplexity" = "Effective number of choices"

Maximum effective number of choices is $|\mathcal{A}|$

Summary of Lecture 6

The **Huffman Algorithm** gives optimal symbol codes:
Merging event adds to code length for children, so
Huffman always merges least probable events first

A complete code implies negative log probabilities: $q_i = 2^{-\ell_i}$.
If the symbols are generated with these probabilities, the symbol code compresses to the entropy. Otherwise the number of extra bits/symbol is given by the **Relative Entropy** or **KL-divergence**: $D_{\mathrm{KL}}(p\,||\,q) = \sum_i p_i \log \frac{p_i}{q_i}$

Gibbs' inequality says $D_{\mathrm{KL}}(p\,||\,q) \geq 0$ with equality only when the distributions are equal.

Jensen's inequality is a useful means to prove several inequalities in Information Theory including (it will turn out) Gibbs' inequality.