Where are we?

Last time . . .
- Talked about decision making under uncertainty
- Looked at utility theory
- Discussed axioms of utility theory
- Described different utility functions
- Introduced decision networks

Today . . .
- **Markov Decision Processes**
Sequential decision problems

- So far we have only looked at one-shot decisions, but decision process are often sequential.
- Example scenario: a 4x3-grid in which agent moves around (fully observable) and obtains utility of +1 or -1 in terminal states.

Actions are somewhat unreliable (in deterministic world, solution would be trivial).
Markov decision processes

- To describe such worlds, we can use a (transition) model \( T(s, a, s') \) denoting the probability that action \( a \) in \( s \) will lead to state \( s' \).
- Model is Markovian: probability of reaching \( s' \) depends only on \( s \) and not on history of earlier states.
- Think of \( T \) as big three-dimensional table (actually a DBN).
- Utility function now depends on environment history.
  - agent receives a reward \( R(s) \) in each state \( s \) (e.g. -0.04 apart from terminal states in our example).
  - (for now) utility of environment history is the sum of state rewards.
- In a sense, stochastic generalisation of search algorithms!
Markov decision processes

- Definition of a Markov Decision Process (MDP):
  - Initial state: $S_0$
  - Transition model: $T(s, a, s')$
  - Utility function: $R(s)$

- Solution should describe what agent does in every state
- This is called policy, written as $\pi$
- $\pi(s)$ for an individual state describes which action should be taken in $s$
- Optimal policy is one that yields the highest expected utility (denoted by $\pi^*$)
Example

- Optimal policies in the 4x3-grid environment
  
  (a) With cost of -0.04 per intermediate state $\pi^*$ is conservative for (3,1)
  
  (b) Different cost induces direct run to terminal state/shortcut at (3,1)/no risk/avoid both exits
MDPs very popular in various disciplines, different algorithms for finding optimal policies

Before we present some of them, let us look at utility functions more closely

We have used sum of rewards as utility of environment history until now, but what are the alternatives?

First question: finite horizon or infinite horizon

Finite means there is a fixed time $N$ after which nothing matters:

$$\forall k \ U_h([s_0, s_1, \ldots, s_{N+k}]) = U_h([s_0, s_1, \ldots, s_N])$$
Optimality in sequential decision problems

- This leads to **non-stationary** optimal policies ($N$ matters)
- With infinite horizon, we get **stationary** optimal policies (time at state doesn't matter)
- We are mainly going to use infinite horizon utility functions
- NOTE: sequences to terminal states can be finite even under infinite horizon utility calculation
- Second issue: how to calculate utility of sequences
- **Stationarity** here is reasonable assumption:

$$s_0 = s'_0 \land [s_0, s_1, s_2 \ldots] \succ [s'_0, s'_1, s'_2, \ldots] \Rightarrow [s_1, s_2 \ldots] \succ [s'_1, s'_2, \ldots]$$
Optimality in sequential decision problems

- Stationarity may look harmless, but there are only two ways to assign utilities to sequences under stationarity assumptions

- **Additive rewards:**

  \[ U_h([s_0, s_1, s_2 \ldots]) = R(s_0) + R(S_1) + R(S_2) + \ldots \]

- **Discounted rewards** (for **discount factor** \(0 \leq \gamma \leq 1\))

  \[ U_h([s_0, s_1, s_2 \ldots]) = R(s_0) + \gamma R(S_1) + \gamma^2 R(S_2) + \ldots \]

- Discount factor makes more distant future rewards less significant

- We will mostly use discounted rewards in what follows
Optimality in sequential decision problems

- Choosing infinite horizon rewards creates a problem
- Some sequences will be infinite with infinite (additive) reward, how do we compare them?
- Solution 1: with discounted rewards the utility is bounded if single-state rewards are

\[ U_h([s_0, s_1, s_2, \ldots]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \sum_{t=0}^{\infty} \gamma^t R_{\text{max}} = R_{\text{max}} / (1 - \gamma) \]

- Solution 2: under \textit{proper policies}, i.e. if agent will eventually visit terminal state, additive rewards are finite
- Solution 3: compare \textit{average reward} per time step
Value iteration

- **Value iteration** is an algorithm for calculating optimal policy in MDPs. *Calculate the utility of each state and then select optimal action based on these utilities.*

- Since discounted rewards seemed to create no problems, we will use

\[
\pi^* = \arg \max_\pi \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t R(s_t) \middle| \pi \right]
\]

as a criterion for optimal policy.
Explaning \( \pi^* = \arg \max_{\pi} E \left[ \sum_{t=0}^{\infty} \gamma^t R(s_t) \right] \)

- Each policy \( \pi \) yields a tree, with root node \( s_0 \), and daughters to a node \( s \) are the possible successor states given the action \( \pi(s) \).
  - \( T(s, a, s') \) gives the probability of traversing an arc from \( s \) to daughter \( s' \).

- \( E \) is computed by:
  (a) For each path \( p \) in the tree, getting the product of the (joint) probability of the path in this tree with its discounted reward, and then
  (b) Summing over all the products from (a)

- So this is just a generalisation of single shot decision theory.
Utilities of states: $U(s) \neq R(s)$!

- $R(s)$ is reward for being in $s$ now.
- By making $U(s)$ the utility of the states that might follow it, $U(s)$ captures long-term advantages from being in $s$.
  
  $U(s)$ reflects what you can do from $s$; $R(s)$ does not.

- States that follow depend on $\pi$. So utility of $s$ given $\pi$ is:

  $$U^\pi(s) = E \left[ \sum_{t=0}^{\infty} \gamma^t R(s_t) \mid \pi, s_0 = s \right]$$

- With this, “true” utility $U(s)$ is $U^{\pi^*}(s)$ (expected sum of discounted rewards if executing optimal policy)
Utilities in our example

- $U(s)$ computed for our example from algorithms to come.
- $\gamma = 1$, $R(s) = -0.04$ for nonterminals.

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Utilities of states

- Given $U(s)$, we can easily determine optimal policy:

$$\pi^*(s) = \arg\max_a \sum_{s'} T(s, a, s') U(s')$$

- Direct relationship between utility of a state and that of its neighbours:

  Utility of a state is immediate reward plus expected utility of subsequent states if agent chooses optimal action

- This can be written as the famous **Bellman equations**:

$$U(s) = R(s) + \gamma \max_a \sum_{s'} T(s, a, s') U(s')$$
The value iteration algorithm

- For $n$ states we have $n$ Bellman equations with $n$ unknowns (utilities of states)
- Value iteration is an iterative approach to solving the $n$ equations.
- Start with arbitrary values and update them as follows:

$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_a \sum_{s'} T(s, a, s') U_i(s')$$

- The algorithm converges to right and unique solution
- Like propagating values through network or utilities
The value iteration algorithm

- Value iteration in our example: evolution of utility values of states
Decision-theoretic agents

- We now have (tediously) gathered all the ingredients to build decision-theoretic agents
- Transition and observation models will be described by a DBN
- They will be augmented by decision and utility nodes to obtain a dynamic DN
- Decisions will be made by projecting forward possible action sequences and choosing the best one
- Practical design for a utility-based agent
Decision-theoretic agents

- Dynamic decision networks look something like this
- General form of everything we have talked about in uncertainty part

\[
\begin{align*}
X_{t-1} \xrightarrow{A_{t-2}} X_t \xrightarrow{A_{t-1}} X_t \xrightarrow{A_{t-1}} X_t \xrightarrow{A_{t+1}} X_{t+1} \xrightarrow{A_{t+1}} X_{t+2} \xrightarrow{A_{t+2}} X_{t+3} \xrightarrow{U_{t+3}} \\
E_{t-1} \xrightarrow{R_{t-1}} E_t \xrightarrow{R_t} E_t \xrightarrow{R_{t+1}} E_{t+1} \xrightarrow{R_{t+2}} E_{t+2} \xrightarrow{E_{t+3}} 
\end{align*}
\]
Summary

- Sequential decision making
- Defined MDPs to model stochastic multi-step decision making processes
- Value iteration and policy iteration algorithms
- Design of decision-theoretic utility-based agents based on DDNs
- Completes our account of reasoning under uncertainty