Outline

- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs
- Making this more efficient
Standard search problem:

- **state** is a ‘black box’ – any data structure that supports successor function, heuristic function and goal test.

CSP:

- **state** is defined by variables $X_i$ with values from domain $D_i$ $(i = 1, \ldots, n)$
- goal test is a set of constraints specifying allowable combinations of values for subsets of variables.
- Simple example of a **formal representation language**.
- Allows useful **general-purpose** algorithms with more power than standard search algorithms.
Example: Map-Colouring

- Variables $WA, NT, Q, NSW, V, SA, T$
- Domains $D_i = \text{red}, \text{green}, \text{blue}$
- Constraints: adjacent regions must have different colours,
  - e.g. $WA \neq NT$,
  - or $(WA, NT)$ in

$\{(\text{red,green}), (\text{red,blue}), (\text{green,red}), (\text{green,blue}), (\text{blue,red}), (\text{blue,green})\}$. 

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Example: Map-Colouring

Solutions are complete and consistent assignments,

- e.g. WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green.
**Constraint graph**

- **Binary CSP**: each constraint relates two variables.
- **Constraint graph**:
  - nodes are variables,
  - arcs (edges) represent constraints relating two nodes each.
Varieties of CSPs

- **Discrete variables:**
  - finite domains:
    - $n$ variables, domain size $d \rightarrow O(d^n)$, complete assignments.
    - e.g. Boolean CSPs, incl. Boolean satisfiability (NP-complete).
  - infinite domains:
    - integers, strings, etc.
    - e.g. job scheduling, variables are start/end days for each job.
    - need a constraint language, e.g.
      \[ \text{StartJob}_1 + 5 \leq \text{StartJob}_3 \]

- **Continuous variables:**
  - e.g. start/end times for Hubble Space Telescope observations.
  - linear constraints solvable in polynomial time by linear programming.
Varieties of constraints

- **Unary** constraints involve a single variable,
  - e.g. $SA \neq \text{green}$.

- **Binary** constraints involve pairs of variables,
  - e.g. $SA \neq WA$.

- **Higher-order** constraints involve 3 or more variables,
  - e.g. crypt-arithmetic column constraints.

- **Global** constraints involve an arbitrary number of variables.
Example: Crypt-arithmetic

Variables: $F, T, U, W, R, O, X_1, X_2, X_3$

Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints: $\text{Alldiff}(F, T, U, W, R, O)$

- $O + O = R + 10 \cdot X_1$
- $X_1 + W + W = U + 10 \cdot X_2$
- $X_2 + T + T = O + 10 \cdot X_3$
- $X_3 = F, \ T \neq 0, \ F \neq 0$. 
Real-world CSPs

- Assignment problems
  - e.g. who teaches what class.
- Timetabling problems.
  - e.g. which class is offered when and where.
- Transportation scheduling.
- Factory scheduling.

Notice that many real-world problems involve real-valued variables.
Let’s start with the straightforward approach, then adapt it. States are defined by the values assigned so far.

- **Initial state**: the empty assignment `{ }`.
- **Successor function**: assign a value to an unassigned variable that does not conflict with current assignment → fail if no legal assignments.
- **Goal test**: the current assignment is complete.

1. This is the same for all CSPs.
2. Every solution appears at depth $n$ with $n$ variables → use depth-first search.
Variable assignments are **commutative**, e.g. [ WA = red then NT = green ] same as [ NT = green then WA = red ].

Only need to consider assignments to a single variable at each node → \( b = d \) and there are \( d^n \) leaves.

Depth-first search for CSPs with single-variable assignments is called **backtracking** search.

Backtracking search is the basic uninformed algorithm for CSPs.

Can solve \( n \)-queens for \( n \approx 25 \).
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
    return BACKTRACK({}, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-Variable(csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment then
            add \{var = value\} to assignment
            inferences ← INERENCE(csp, var, value)
            if inferences ≠ failure then
                add inferences to assignment
                result ← BACKTRACK(assignment, csp)
                if result ≠ failure then
                    return result
                remove \{var = value\} and inferences from assignment
        return failure
Backtracking example
Backtracking example
Backtracking example
Backtracking example
Which nodes can be eliminated on symmetry grounds?
General-purpose methods can give huge gains in speed:

- Which variable should be assigned next?
  - SELECT-UNASSIGNED-VARIABLE
- Then, in what order should its values be tried?
  - ORDER-DOMAIN-VALUES
- What inferences should be performed at each step of the search?
  - INFERENCE
- Can we detect inevitable failure early?
Most constrained variable

\[ \text{var} \leftarrow \text{SELECT-UNASSIGNED-VARIABLE}(csp) \]

- Most constrained variable: choose the variable with the fewest legal values.

- a.k.a. minimum-remaining-values (MRV) heuristic.
Most constraining variable

- Tie-breaker among most constrained variables.
- Most constraining variable:
  - choose the variable with the most constraints on remaining variables – thus reducing branching.
- a.k.a. degree heuristic
Given a variable, choose the least constraining value:

- the one that rules out the fewest values in the remaining variables.

Combining these heuristics makes 1000 queens feasible.
Inference: Forward checking

Idea:

- Keep track of remaining legal values for unassigned variables.
- Terminate search when any variable has no legal values.

![Diagram of Australia showing states and colors]
Forward checking

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Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn’t provide early detection for all failures:

- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally.
Arc consistency

- Simplest form of propagation makes each arc consistent.
- \( X \rightarrow Y \) is consistent iff
- for every value \( x \) of in the domain of \( X \) there is some allowed \( y \) in the domain of \( Y \).
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![Diagram of Arc Consistency](image)
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Arc consistency detects failure earlier than forward checking.
Can be run as a preprocessor or after each assignment.
Arc consistency algorithm AC-3

```plaintext
function AC-3(csp) returns false if an inconsistency is found and true otherwise
inputs: csp, a binary CSP with components (X, D, C)
local variables: queue, a queue of arcs, initially all the arcs in csp

while queue is not empty do
  (X_i, X_j) ← REMOVE-FIRST(queue)
  if REVISE(csp, X_i, X_j) then
    if size of D_i = 0 then return false  # No consistent value left for X_i so fail
    for each X_k in X_i.NEIGHBORS - {X_j} do
      add (X_k, X_i) to queue  # Since revision occurred, add all neighbours of X_i for consideration (or reconsideration)
  return true
```

```plaintext
function REVISE(csp, X_i, X_j) returns true iff we revise the domain of X_i
revised ← false
for each x in D_i do
  if no value y in D_j allows (x, y) to satisfy the constraint between X_i and X_j then
    delete x from D_i
    revised ← true
return revised
```

- Time complexity: $O(cd^3)$, where $d$ is maximum size of each domain and $c$ is the number of binary constraints (arcs).
- Space complexity $O(c)$
CSPs are a special kind of problem:

- States defined by values of a fixed set of variables
- Goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g. arc consistency) does additional work to constrain values and detect inconsistencies