Inf2D 07: Effective Propositional Inference

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Credits: The content of this lecture was prepared by Michael Rovatsos and follows R&N
Two families of efficient algorithms for propositional inference:

- Complete backtracking search algorithms
  - DPLL algorithm (Davis, Putnam, Logemann, Loveland)
- Incomplete local search algorithms
  - WalkSAT algorithm
DPLL and WalkSAT manipulate formulae in **conjunctive normal form (CNF)**.

**Sentence** is formula whose satisfiability is to be determined.

- conjunction of clauses.

**Clause** is **disjunction** of literals

**Literal** is proposition or **negated proposition**

**Example:** $(A, \neg B), (B, \neg C)$ representing $(A \lor \neg B) \land (B \lor \neg C)$
Conversion to CNF

\[ B_{1,1} \iff (P_{1,2} \lor P_{2,1}) \]

1. Eliminate \( \iff \) replacing \( \alpha \iff \beta \) by \( (\alpha \implies \beta) \land (\beta \implies \alpha) \)
   \( (B_{1,1} \implies (P_{1,2} \lor P_{2,1})) \land ((P_{1,2} \lor P_{2,1}) \implies B_{1,1}) \)

2. Eliminate \( \implies \), replacing \( \alpha \implies \beta \) by \( \neg \alpha \lor \beta \)
   \( (\neg B_{1,1} \lor (P_{1,2} \lor P_{2,1})) \land (\neg (P_{1,2} \lor P_{2,1}) \lor B_{1,1}) \)

3. Move \( \neg \) inwards using de Morgan’s rules
   \( (\neg B_{1,1} \lor (P_{1,2} \lor P_{2,1})) \land ((\neg P_{1,2} \land \neg P_{2,1}) \lor B_{1,1}) \)
   possibly also eliminating double-negation: replacing \( \neg (\neg \alpha) \) by \( \alpha \)

4. Apply distributivity law (\( \lor \) over \( \land \)) and flatten:
   \( (\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1}) \)
The DPLL algorithm

Determine if an input propositional logic sentence (in CNF) is **satisfiable**.

**Improvements** over truth table enumeration:

1. Early termination
2. Pure symbol heuristic
3. Unit clause heuristic
A clause is true if one of its literals is true,
  e.g. if $A$ is true then $(A \vee \neg B)$ is true.
A sentence is false if any of its clauses is false,
  e.g. if $A$ is false and $B$ is true then $(A \vee \neg B)$ is false, so sentence containing it is false.
Pure symbol heuristic

- **Pure symbol**: always appears with the same “sign” or polarity in all clauses.
  - e.g., in the three clauses \((A \lor \neg B), (\neg B \lor \neg C), (C \lor A)\)
    - \(A\) and \(B\) are pure, \(C\) is impure.

- Make **literal** containing a pure symbol true.
  - e.g. (for satisfiability) Let \(A\) and \(\neg B\) both be true
Unit clause: only one literal in the clause, e.g. \((A)\)

- The only literal in a unit clause must be true.
  - e.g. \(A\) must be true.
- Also includes clauses where all but one literal is false,
  - e.g. \((A, B, C)\) where \(B\) and \(C\) are false since it is equivalent to \((A, \text{false}, \text{false})\) i.e. \((A)\).
The DPLL algorithm

```
function DPLL-SATISFIABLE?(s) returns true or false
    inputs: s, a sentence in propositional logic
    clauses ← the set of clauses in the CNF representation of s
    symbols ← a list of the proposition symbols in s
    return DPLL(clauses, symbols, [])

function DPLL(clauses, symbols, model) returns true or false
    if every clause in clauses is true in model then return true
    if some clause in clauses is false in model then return false
    P, value ← FIND-PURE-SYMBOL(symbols, clauses, model)
    if P is non-null then return DPLL(clauses, symbols−P, [P = value|model])
    P, value ← FIND-UNIT-CLAUSE(clauses, model)
    if P is non-null then return DPLL(clauses, symbols−P, [P = value|model])
    P ← FIRST(symbols); rest ← REST(symbols)
    return DPLL(clauses, rest, [P = true|model]) or
           DPLL(clauses, rest, [P = false|model])
```
Tautology Deletion (Optional)

- **Tautology**: both a proposition and its negation in a clause.
  - e.g. \((A, B, \neg A)\)
- Clause bound to be true.
  - e.g. whether \(A\) is true or false.
  - Therefore, can be deleted.
Mid-Lecture Exercise

- Apply DPLL heuristics to the following sentence:

\[(S_{2,1}), (\neg S_{1,1}), (\neg S_{1,2}), (\neg S_{2,1}, W_{2,2}), (\neg S_{1,1}, W_{2,2}), (\neg S_{1,2}, W_{2,2}), (\neg W_{2,2}, S_{2,1}, S_{1,1}, S_{1,2})].\]

- Use case splits if model not found by these heuristics.
Pure symbol heuristic:
- No literal is pure.

Unit clause heuristic:
- $S_{2,1}$ is true; $S_{1,1}$ and $S_{1,2}$ are false.

Early termination heuristic:
- $(\neg S_{1,1}, W_{2,2}), (\neg S_{1,2}, W_{2,2})$ are both true.
- $(\neg W_{2,2}, S_{2,1}, S_{1,1}, S_{1,2})$ is true.

Unit clause heuristic:
- $\neg S_{2,1}$ is false, so $(\neg S_{2,1}, W_{2,2})$ becomes unit clause.
- $W_{2,2}$ must be true.
The WalkSAT algorithm

- Incomplete, local search algorithm
- Evaluation function: The min-conflict heuristic of minimizing the number of unsatisfied clauses
- Balance between greediness and randomness
The WalkSAT algorithm

```plaintext
function WALKSAT(clauses, p, max-flips) returns a satisfying model or failure
inputs: clauses, a set of clauses in propositional logic
        p, the probability of choosing to do a “random walk” move
        max-flips, number of flips allowed before giving up

model ← a random assignment of true/false to the symbols in clauses
for i = 1 to max-flips do
    if model satisfies clauses then return model
    clause ← a randomly selected clause from clauses that is false in model
    with probability p flip the value in model of a randomly selected symbol
    from clause
    else flip whichever symbol in clause maximizes the number of satisfied clauses
return failure
```

Algorithm checks for satisfiability by randomly flipping the values of variables
Consider random 3-CNF sentences: 3SAT problem

Example:

\[(\neg D \lor \neg B \lor C) \land (B \lor \neg A \lor \neg C) \land (\neg C \lor \neg B \lor E) \land (E \lor \neg D \lor B) \land (B \lor E \lor \neg C)\]

- \(m\): number of clauses
- \(n\): number of symbols

Hard problems seem to cluster near \(m/n = 4.3\) (critical point)
Hard satisfiability problems

![Graph showing the probability of satisfiability as a function of the clause/symbol ratio $m/n$.](image)

- $Pr(satisfiable)$
- $Clause/symbol ratio m/n$

The graph illustrates the decrease in the probability of a formula being satisfiable as the ratio $m/n$ increases, indicating the transition from easy to hard satisfiability problems.
Median runtime for 100 satisfiable random 3-CNF sentences, $n = 50$
A wumpus-world agent using propositional logic:

\(-P_{1,1}\)
\(-W_{1,1}\)

\(B_{x,y} \iff (P_{x,y+1} \lor P_{x,y-1} \lor P_{x+1,y} \lor P_{x-1,y})\)

\(S_{x,y} \iff (W_{x,y+1} \lor W_{x,y-1} \lor W_{x+1,y} \lor W_{x-1,y})\)

\(W_{1,1} \lor W_{1,2} \lor \cdots \lor W_{4,4}\)

\(-W_{1,1} \lor -W_{1,2}\)

\(-W_{1,1} \lor -W_{1,3}\)

\(\cdots\)

\(\Rightarrow 64\) distinct proposition symbols, 155 sentences
function \textsc{Hybrid-Wumpus-Agent} (percept) returns an action
\begin{itemize}
\item[] \textbf{inputs:} percept, a list, [stench, breeze, glitter, bump, scream]
\item[] \textbf{persistent:} \( KB \), a knowledge base, initially the atemporal “wumpus physics”
\item[] \hspace{1cm} \( t \), a counter, initially 0, indicating time
\item[] \hspace{1cm} \( plan \), an action sequence, initially empty
\end{itemize}

\textsc{Tell}(\( KB \), \textsc{Make-Percept-Sentence}(percept, \( t \)))

Tell the \( KB \) the temporal “physics” sentences for time \( t \)

\( \text{safe} \leftarrow \{[x,y] : \text{Ask}(KB, OK^t_{x,y}) = true\} \)

\begin{itemize}
\item[] if \( \text{Ask}(KB, \text{Glitter}^t) = true \) then
\item[] \hspace{1cm} \( \text{plan} \leftarrow \text{[Grab]} + \text{Plan-Route}(current, \{[1,1]\}, \text{safe}) + \text{[Climb]} \)
\end{itemize}

\begin{itemize}
\item[] if \( \text{plan is empty} \) then
\item[] \hspace{1cm} \( \text{unvisited} \leftarrow \{[x,y] : \text{Ask}(KB, I^t_{x,y}) = false \text{ for all } t' \leq t \} \)
\item[] \hspace{1cm} \( \text{plan} \leftarrow \text{Plan-Route}(\text{current}, \text{unvisited} \cap \text{safe}, \text{safe}) \)
\end{itemize}

\begin{itemize}
\item[] if \( \text{plan is empty and Ask}(KB, \text{HaveArrow}^t) = true \) then
\item[] \hspace{1cm} \( \text{possible_wumpus} \leftarrow \{[x,y] : \text{Ask}(KB, \neg W_{x,y}) = false \} \)
\item[] \hspace{1cm} \( \text{plan} \leftarrow \text{Plan-Shot}(\text{current}, \text{possible_wumpus}, \text{safe}) \)
\end{itemize}

\begin{itemize}
\item[] if \( \text{plan is empty} \) then // no choice but to take a risk
\item[] \hspace{1cm} \( \text{notUnsafe} \leftarrow \{[x,y] : \text{Ask}(KB, \neg OK^t_{x,y}) = false \} \)
\item[] \hspace{1cm} \( \text{plan} \leftarrow \text{Plan-Route}(\text{current}, \text{unvisited} \cap \text{notUnsafe}, \text{safe}) \)
\end{itemize}

\begin{itemize}
\item[] if \( \text{plan is empty} \) then
\item[] \hspace{1cm} \( \text{plan} \leftarrow \text{Plan-Route}(\text{current}, \{[1,1]\}, \text{safe}) + \text{[Climb]} \)
\end{itemize}

\( \text{action} \leftarrow \text{Pop(plan)} \)

\textsc{Tell}(\( KB \), \textsc{Make-Action-Sentence}(\( action \), \( t \)))

\( t \leftarrow t+1 \)

return action
function PLAN-ROUTE(current, goals, allowed) returns an action sequence
inputs: current, the agent’s current position
goals, a set of squares; try to plan a route to one of them
allowed, a set of squares that can form part of the route

problem ← ROUTE-PROBLEM(current, goals, allowed)
return A*GRAPH-SEARCH(problem)

We will look at this later on.
We need more!

- **Effect axioms:**
  \[ L_{1,1}^0 \land \text{FacingEast}^0 \land \text{Forward}^0 \Rightarrow L_{2,1}^1 \land \neg L_{11}^1 \]

- We need extra axioms about the world.
- **Representational frame problem**
  - **Frame axioms:**
    \begin{align*}
    \text{Forward}^t &\Rightarrow (\text{HaveArrow}^t \Leftrightarrow \text{HaveArrow}^{t+1}) \\
    \text{Forward}^t &\Rightarrow (\text{WumpusAlive}^t \Leftrightarrow \text{WumpusAlive}^{t+1})
    \end{align*}
- **Inferential frame problem**
  - **Successor-state axioms:**
    \[ \text{HaveArrow}^{t+1} \Leftrightarrow (\text{HaveArrow}^t \land \neg \text{Shoot}^t) \]
Expressiveness limitation of propositional logic

- KB contains “physics” sentences for every single square
- For every time $t$ and every location $[x, y]$,

$$L_{x,y}^t \land \text{FacingRight}^t \land \text{Forward}^t \Rightarrow L_{x+1,y}^{t+1}$$

- Rapid proliferation of clauses
Logical agents apply inference to a knowledge base to derive new information and make decisions.

- Two algorithms: DPLL & WalkSAT
- Hard satisfiability problems
- Applications to Wumpus World.
- Propositional logic lacks expressive power