Inf2b Learning and Data

Lecture 12: Single layer Neural Networks (1)

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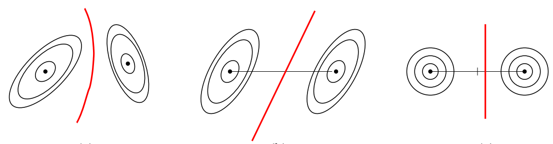
Today's Schedule

1. Discriminant functions (recap)
2. Decision boundary of linear discriminants
3. Discriminants for multiple classes
4. Training of linear discriminant functions
5. Perceptron

Discriminant functions (recap)

$$y_c(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_c)^T\Sigma_c^{-1}(\mathbf{x}-\boldsymbol{\mu}_c) - \frac{1}{2}\ln|\Sigma_c| + \ln P(c)$$
$$= -\frac{1}{2}\mathbf{x}^T\Sigma_c^{-1}\mathbf{x} + \boldsymbol{\mu}_c^T\Sigma_c^{-1}\mathbf{x} - \frac{1}{2}\boldsymbol{\mu}_c^T\Sigma_c^{-1}\boldsymbol{\mu}_c - \frac{1}{2}\ln|\Sigma_c| + \ln P(c$$



(a) (b) (c)

Linear discriminants for a 2-class problem

$$y_1(\mathbf{x}) = \mathbf{w}_1^T\mathbf{x} + w_{10}$$
$$y_2(\mathbf{x}) = \mathbf{w}_2^T\mathbf{x} + w_{20}$$

Combined discriminant function:

$$y(\mathbf{x}) = y_1(\mathbf{x}) - y_2(\mathbf{x}) = (\mathbf{w}_1 - \mathbf{w}_2)^T\mathbf{x} + (w_{10} - w_{20})$$
$$= \mathbf{w}^T\mathbf{x} + w_0$$

Decision:

$$C = \begin{cases} 1, & \text{if } y(\mathbf{x}) \geq 0, \\ 2, & \text{if } y(\mathbf{x}) < 0 \end{cases}$$

Decision boundary of linear discriminants

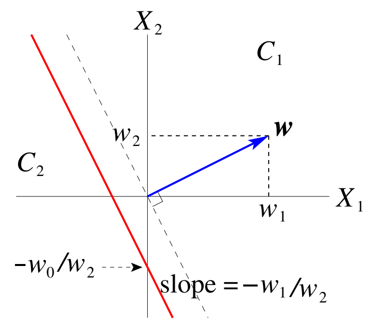
- Decision boundary:
  $\mathbf{w}^T\mathbf{x} + w_0 = 0$

Dimension	Decision boundary	
2 :	line	$w_1x_1 + w_2x_2 + w_0 = 0$
3 :	plane	$w_1x_1 + w_2x_2 + w_3x_3 + w_0 = 0$
d :	hyperplane	$(\sum_{i=1}^{d} w_ix_i) + w_0 = 0$

NB: $\mathbf{w}$ is a normal vector to the hyperplane

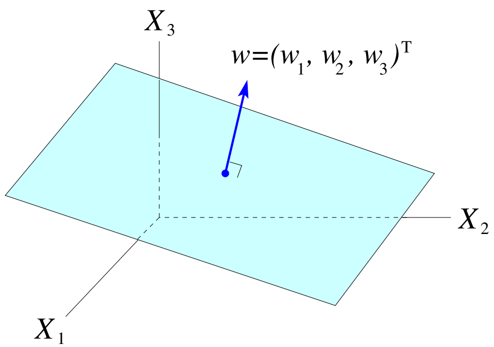
Decision boundary of linear discriminant (2D)

$w_1x_1 + w_2x_2 + w_0 = 0$



Decision boundary of linear discriminant (3D)

$w_1x_1 + w_2x_2 + w_3x_3 + w_0 = 0$



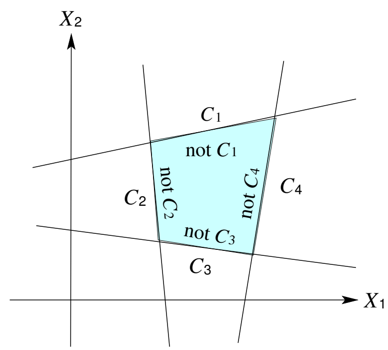
Discriminants for multiple classes ($K > 2$)

- One-versus-the-rest classifiers $\cdots$ K classifiers
  E.g. $K = 4$

$$y_1(\mathbf{x}) = \mathbf{w}_1^T\mathbf{x} + w_{10}$$
$$y_2(\mathbf{x}) = \mathbf{w}_2^T\mathbf{x} + w_{20}$$
$$y_3(\mathbf{x}) = \mathbf{w}_3^T\mathbf{x} + w_{30}$$
$$y_4(\mathbf{x}) = \mathbf{w}_4^T\mathbf{x} + w_{40}$$

- What if $\forall i :\ y_i(\mathbf{x}) < 0$?

Discriminants for multiple classes ($K > 2$)



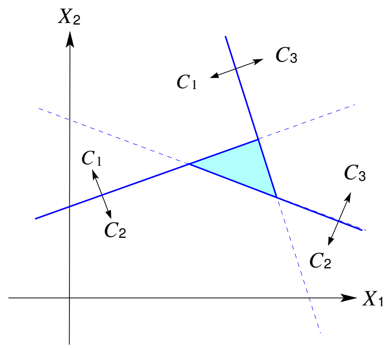
Multi-class discriminants (one-vs-one)

- One-versus-one classifiers : $\cdots$ $K(K-1)/2$ classifiers
  E.g. $K = 3$

$$y_{12}(\mathbf{x}) = \mathbf{w}_{12}^T\mathbf{x} + w_{12,0}$$
$$y_{23}(\mathbf{x}) = \mathbf{w}_{23}^T\mathbf{x} + w_{23,0}$$
$$y_{31}(\mathbf{x}) = \mathbf{w}_{31}^T\mathbf{x} + w_{31,0}$$

- What if $y_{12}(\mathbf{x}) < 0,\ y_{23}(\mathbf{x}) < 0,\ y_{31}(\mathbf{x}) < 0$?

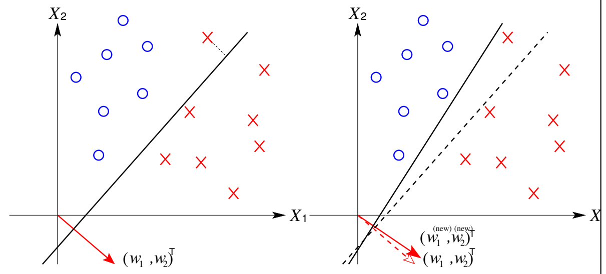
Multi-class discriminants (one-vs-one)



Training of linear discriminant functions

- A discriminant for a two-class problem:

$$y(\mathbf{x}) = y_1(\mathbf{x}) - y_2(\mathbf{x}) = (\mathbf{w}_1 - \mathbf{w}_2)^T\mathbf{x} + (w_{10} - w_{20})$$
$$= \mathbf{w}^T\mathbf{x} + w_0$$



Error correction algorithm

$$a(\dot{\mathbf{x}}) = \mathbf{w}^T\mathbf{x} + w_0 \ = \ \dot{\mathbf{w}}^T\dot{\mathbf{x}}$$

where $\dot{\mathbf{w}} = (w_0, \mathbf{w}^T)^T,\ \dot{\mathbf{x}} = (1, \mathbf{x}^T)^T$

Let's just use $\mathbf{w}$ *and* $\mathbf{x}$ *to denote* $\dot{\mathbf{w}}$ *and* $\dot{\mathbf{x}}$ *from now on!*

$$y(\mathbf{x}) = g(\,a(\mathbf{x})\,) \ = \ g(\,\mathbf{w}^T\mathbf{x}\,)$$

$$\text{where } g(a) = \begin{cases} +1, & \text{if } a \geq 0, \\ -1, & \text{if } a < 0 \end{cases}$$

- Training set : $D = \{(\mathbf{x}^{(1)}, t^{(1)}), \ldots, (\mathbf{x}^{(N)}, t^{(N)})\}$
  where $t^{(i)} \in \{-1, +1\}$
- If $y(\mathbf{x}^{(i)}) = -1$ for $t^{(i)} = +1$,
  $\mathbf{w}^{(\text{new})} \leftarrow \mathbf{w} + \eta\,\mathbf{x}^{(i)} \qquad (\eta > 0)$
- If $y(\mathbf{x}^{(i)}) = +1$ for $t^{(i)} = -1$,
  $\mathbf{w}^{(\text{new})} \leftarrow \mathbf{w} - \eta\,\mathbf{x}^{(i)} \qquad (\eta > 0)$

The Perceptron criterion

- The number of misclassification:

$$E = \sum_{i=1}^{N} |\,t^{(i)} - y(\mathbf{x}^{(i)})\,|\,/\,2$$

- Instead of calculating the number of misclassification, use the following error measure — Perceptron criterion:

$$E_p(\mathbf{w}) = -\sum_{n\in\mathcal{M}} \mathbf{w}^T\mathbf{x}^{(n)}\,t^{(n)}$$

where $\mathcal{M}$: a set of all misclassified samples.

No misclassification $\quad E_p(\mathbf{w}) = 0$
Misclassification $\quad E_p(\mathbf{w}) > 0$

$$\nabla E_p(\mathbf{w}) = (\partial E_p/\partial w_i) = -\nabla \sum_{n\in\mathcal{M}} \mathbf{w}^T\mathbf{x}^{(n)}\,t^{(n)}$$
$$= -\sum_{n\in\mathcal{M}} \mathbf{x}^{(n)}\,t^{(n)}$$

The Perceptron learning algorithm

Batch Perceptron algorithm:

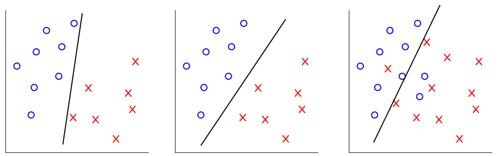
$\mathcal{M} \leftarrow \phi$
for $i = 1, \ldots, N$
  if $g(\,\mathbf{w}^T\mathbf{x}^{(i)}\,) \neq t^{(i)}$
    $\mathcal{M} \leftarrow \{\mathcal{M}, i\}$
$\mathbf{w}^{(\text{new})} \leftarrow \mathbf{w} + \eta \sum_{n\in\mathcal{M}} \mathbf{x}^{(n)}\,t^{(n)}$

Incremental (online) Perceptron algorithm:

for $i = 1, \ldots, N$
  if $g(\,\mathbf{w}^T\mathbf{x}^{(i)}\,) \neq t^{(i)}$
    $\mathbf{w}^{(\text{new})} \leftarrow \mathbf{w} + \mathbf{x}^{(i)}\,t^{(i)}$

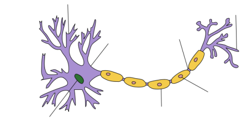
What about convergence?
The Perceptron learning algorithm terminate if training samples are linearly separable.

Linearly separable vs linearly non-separable

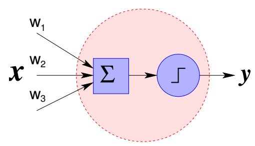


(a–1) (a–2) (b)
Linearly separable Linearly non–separable

Background of Perceptron



(http://en.wikipedia.org/wiki/File:Neuron_Hand-tuned.svg)

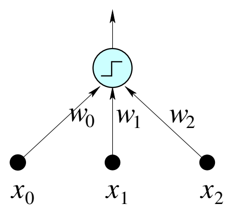


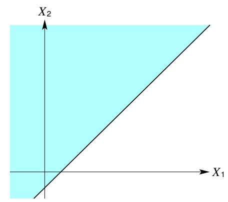
(a) function unit

1940s	Warren McCulloch and Walter Pitts : 'threshold logic'
	Donald Hebb : 'Hebbian learning'
1957	Frank Rosenblatt : 'Perceptron'

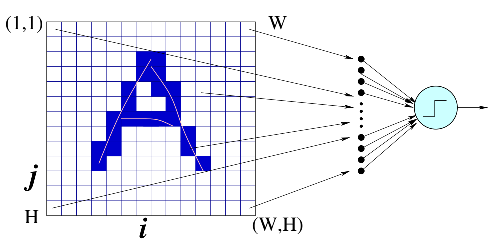
Perceptron architectures and decision boundaries

$$y(\mathbf{x}) = g(\,\mathbf{w}^T\mathbf{x}\,)$$





Character recognition by Perceptron



Perceptron as a logical function

NOT

x_1	y
0	1
1	0

OR

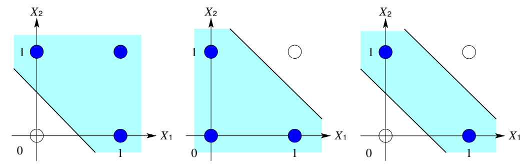
x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	1

NAND

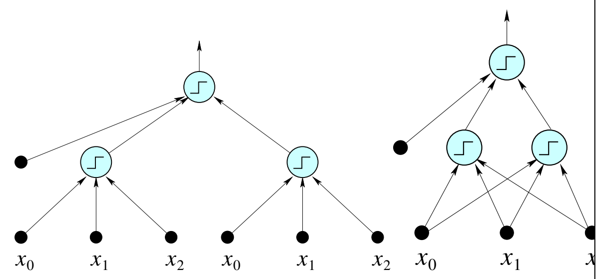
x_1	x_2	y
0	0	1
0	1	1
1	0	1
1	1	0

EXOR

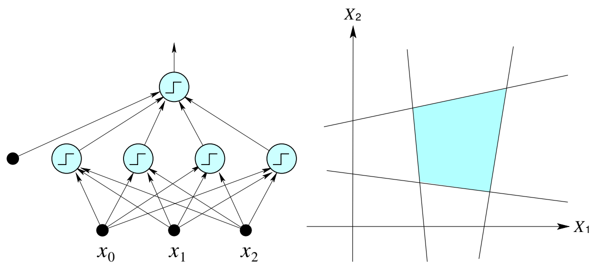
x_1	x_2	y
0	0	0
0	1	1
1	0	1
1	1	0



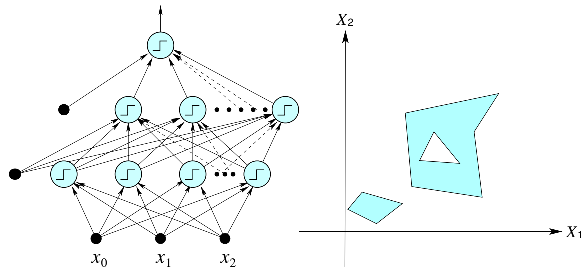
Perceptron architectures and decision boundaries



Perceptron architectures and decision boundaries



Perceptron architectures and decision boundaries



Problems with Perceptron

- Non convergence if the training data are linearly non-separable
- Difficulty training multiple-layer Perceptrons
- Piecewise-linear decision boundaries

Summary

- Decision boundaries of linear discriminant functions
- Discriminant functions for multiple classes
- Training discriminant functions directly
- Perceptrons