Inf 2B: Asymptotic notation and Algorithms

Lecture 2B of ADS thread

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Reminder of Asymptotic Notation

Let $f, g : \mathbb{N} \to \mathbb{R}$ be functions. We say that:

- *f is* $O(g)$ if there is some $n_0 \in \mathbb{N}$ and some $c > 0 \in \mathbb{R}$ such that for all $n \geq n_0$ we have

$$0 \leq f(n) \leq c\,g(n).$$

- *f is* $\Omega(g)$ if there is an $n_0 \in \mathbb{N}$ and $c > 0$ in $\mathbb{R}$ such that for all $n \geq n_0$ we have

$$f(n) \geq c\,g(n) \geq 0.$$

- ***f** is* $\Theta(g)$, or *f has the same asymptotic growth rate* as g, if f is $O(g)$ and $\Omega(g)$.

Worst-case (and best-case) running-time

We almost always work with Worst-case running time in Inf2B:

Definition

The *(worst-case) running time* of an algorithm A is the function $T_{\mathsf{A}} : \mathbb{N} \to \mathbb{N}$ where $T_{\mathsf{A}}(n)$ is the maximum number of computation steps performed by A on an input of size n.

Definition

The *(best-case) running time* of an algorithm A is the function $B_{\mathsf{A}} : \mathbb{N} \to \mathbb{N}$ where $B_{\mathsf{A}}(n)$ is the minimum number of computation steps performed by A on an input of size n.

We only use Best-case for explanatory purposes.

Asymptotic notation for Running-time

How do we apply O, Ω, Θ to analyse the running-time of an algorithm A?

Possible approach:

- We analyse A to obtain the worst-case running time function $T_{\mathsf{A}}(n)$.
- We then go on to derive upper and lower bounds on (the growth rate of) $T_{\mathsf{A}}(n)$, in terms of $O(\cdot), \Omega(\cdot)$.

In fact we use asymptotic notation with the analysis, much simpler (no need to give names to constants, takes care of low level detail that isn't part of the big picture).

- We aim to have matching $O(\cdot), \Omega(\cdot)$ bounds hence have a $\Theta(\cdot)$ bound.
- Not always possible, even for apparently simple algorithms.

Example

algA(A,r,s)
1. **if** $r < s$ **then**
2. **for** $i \leftarrow r$ **to** s **do**
3. **for** $j \leftarrow i$ **to** s **do**
4. $m \leftarrow \lfloor \frac{i+j}{2} \rfloor$
5. algB$(A, i, m-1)$
6. algB(A, m, j)
7. $m \leftarrow \lfloor \frac{r+s}{2} \rfloor$
8. algA$(A, r, m-1)$
9. algA(A, m, s)

algB(A,r,s)
1. **if** $A[r] < A[s]$ **then**
2. swap $A[r]$ with $A[s]$
3. **if** $r < s - r$ **then**
4. algA$(A, r, s - r)$

linSearch

Input: Integer array A, integer k being searched.
Output: The least index i such that $A[i] = k$.

Algorithm linSearch(A, k)

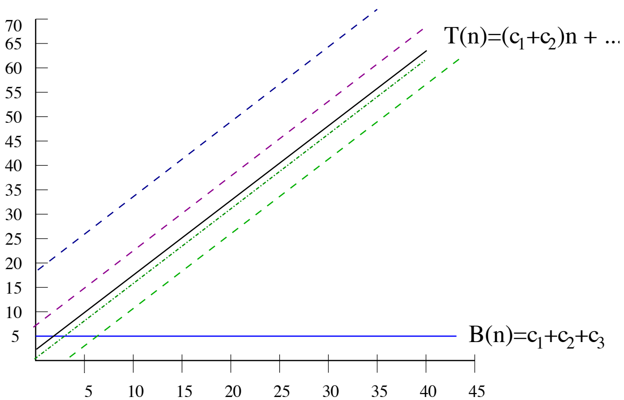
1. **for** $i \leftarrow 0$ **to** $A.length - 1$ **do**
2. **if** $A[i] = k$ **then**
3. **return** i
4. **return** -1

(Lecture Note 1) Worst-case running time $T_{\text{linSearch}}(n)$ satisfies

$$(c_1 + c_2)n + \min\{c_3, c_1 + c_4\} \le T_{\text{linSearch}}(n) \le (c_1 + c_2)n + \max\{c_3, c_1 + c_4\}.$$

Best-case running time satisfies $B_{\text{linSearch}}(n) = c_1 + c_2 + c_3$.

Picture of $T_{\text{linSearch}}(n)$, $B_{\text{linSearch}}(n)$



$T_{\text{linSearch}}(n) = O(n)$

Proof.
From Lecture Note 1 we have

$$T_{\text{linSearch}}(n) \le (c_1 + c_2) \cdot n + \max\{c_3, (c_1 + c_4)\}.$$

Take $n_0 = \max\{c_3, (c_1 + c_4)\}$, $c = c_1 + c_2 + 1$.
Then for every $n \ge n_0$, we have

$$\begin{aligned} T_{\text{linSearch}}(n) &\le (c_1 + c_2)n + n_0 \\ &\le (c_1 + c_2 + 1)n = cn. \end{aligned}$$

Hence $T_{\text{linSearch}}(n) = O(n)$. $\square$

$T_{\text{linSearch}}(n) = \Omega(n)$

We know $T_{\text{linSearch}}(n) = O(n)$.
Also true: $T_{\text{linSearch}}(n) = O(n \lg(n))$, $T_{\text{linSearch}}(n) = O(n^2)$.

Is $T_{\text{linSearch}}(n) = O(n)$ the best we can do?

YES, because ...

$T_{\text{linSearch}}(n) = \Omega(n)$.

Proof.
$T_{\text{linSearch}}(n) \geq (c_1 + c_2)n$, because all c_i are positive.
Take $n_0 = 1$ and $c = c_1 + c_2$ in defn of Ω. $\square$

$$T_{\text{linSearch}}(n) = \Theta(n).$$

Misconceptions/Myths about O and Ω

MISCONCEPTION 1

If we can show $T_A(n) = O(f(n))$ for some function $f : \mathbb{N} \to \mathbb{R}$, then the running time of A on inputs of size n is bounded by $f(n)$ for sufficiently large n.

FALSE: Only guaranteed an upper bound of $cf(n)$, for some constant $c > 0$.

Example: Consider linSearch. We could have shown $T_{\text{linSearch}} = O(\frac{1}{2}(c_1 + c_2)n)$ (or $O(\alpha n)$, for any constant $\alpha > 0$) *exactly* as we showed $T_{\text{linSearch}}(n) = O(n)$ **but** ...
the worst-case for linSearch is greater than $\frac{1}{2}(c_1 + c_2)n$.

Misconceptions/Myths about O and Ω

MISCONCEPTION 2

Because $T_A(n) = O(f(n))$ implies a $c\,f(n)$ upper bound on the running-time of A for *all* inputs of size n, then $T_A(n) = \Omega(g(n))$ implies a similar lower bound on the running-time of A for *all* inputs of size n.

FALSE: If $T_A(n) = \Omega(g(n))$ for some $g : \mathbb{N} \to \mathbb{R}$, then there is some constant $c' > 0$ such that $T_A(n) \geq c'\,g(n)$ for all sufficiently large n.
But A can be much faster than $T_A(n)$ on other inputs of length n that are not worst-case! No lower bound on *general* inputs of size n. linSearch graph is an example.

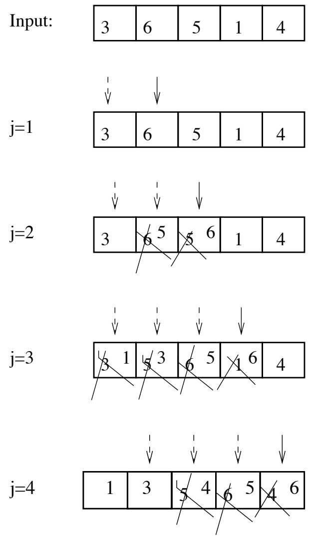
Insertion Sort

Input: An integer array A
Output: Array A sorted in non-decreasing order

Algorithm insertionSort(A)

```
for j ← 1 to A.length − 1 do
    a ← A[j]
    i ← j − 1
    while i ≥ 0 and A[i] > a do
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← a
```

Example: Insertion Sort



Big-O for $T_{\text{insertionSort}}(n)$

Algorithm insertionSort(A)

1. **for** $j \leftarrow 1$ **to** $A.length - 1$ **do**
2. $\quad a \leftarrow A[j]$
3. $\quad i \leftarrow j - 1$
4. $\quad$ **while** $i \geq 0$ and $A[i] > a$ **do**
5. $\qquad A[i+1] \leftarrow A[i]$
6. $\qquad i \leftarrow i - 1$
7. $\quad A[i+1] \leftarrow a$

Line 1 $O(1)$ time, *executed* $A.length - 1 = n - 1$ times.

Lines 2,3,7 $O(1)$ time each, *executed* $n - 1$ times.

Lines 4,5,6 $O(1)$-time, executed together as **for**-loop. No. of executions depends on **for**-test, j.
For fixed j, **for**-loop at 4. takes *at most* j iterations.

Algorithm insertionSort(A)

1. **for** $j \leftarrow 1$ **to** $A.length - 1$ **do**
2. $\quad a \leftarrow A[j]$
3. $\quad i \leftarrow j - 1$
4. $\quad$ **while** $i \geq 0$ and $A[i] > a$ **do**
5. $\qquad A[i+1] \leftarrow A[i]$
6. $\qquad i \leftarrow i - 1$
7. $\quad A[i+1] \leftarrow a$

For a fixed j, lines 2-7 take at most

$$\begin{aligned} & O(1)+O(1)+O(1)+O(j)+O(j)+O(j)+O(1) \\ &= O(1)+O(j) \\ &= O(1)+O(n) \\ &= O(n). \end{aligned}$$

There are $n-1$ different j-values. Hence

$$T_{\text{insertionSort}}(n) = (n-1)O(n) = O(n)O(n) = O(n^2).$$

$T_{\text{insertionSort}}(n) = \Omega(n^2)$

Harder than $O(n^2)$ bound.
Focus on a **BAD** instance of size n:
Take input instance $\langle n, n-1, n-2, \ldots, 2, 1 \rangle$.

- For every $j = 1 \ldots, n-1$, insertionSort uses j executions of line 5 to insert $A[j]$.

Then

$$\begin{aligned} T_{\text{insertionSort}}(n) &\geq \sum_{j=1}^{n-1} c\, j \\ &= c \sum_{j=1}^{n-1} j \;=\; c\frac{n(n-1)}{2}. \end{aligned}$$

So $T_{\text{insertionSort}}(n) = \Omega(n^2)$ and $T_{\text{insertionSort}}(n) = \Theta(n^2)$.

"Typical" asymptotic running times

- $\Theta(\lg n)$ (logarithmic),
- $\Theta(n)$ (linear),
- $\Theta(n \lg n)$ (n-log-n),
- $\Theta(n^2)$ (quadratic),
- $\Theta(n^3)$ (cubic),
- $\Theta(2^n)$ (exponential).

Further Reading

- Lecture notes 2 from last week.
- If you have Goodrich & Tamassia [GT]:
  All of the chapter on "Analysis Tools" (especially the "Seven functions" and "Analysis of Algorithms" sections).
- If you have [CLRS]:
  Read chapter 3 on "Growth of Functions."