# Inf 2B: Graphs, BFS, DFS

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# Directed and Undirected Graphs

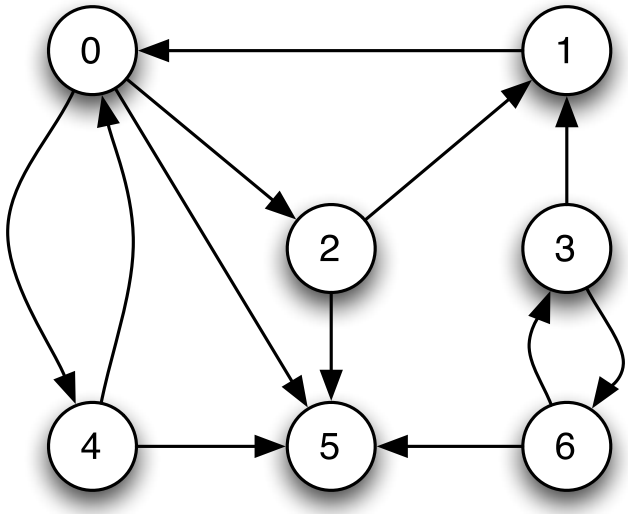
- A *graph* is a mathematical structure consisting of a set of *vertices* and a set of *edges* connecting the vertices.
- Formally: $G = (V, E)$, where $V$ is a set and $E \subseteq V \times V$.
- For edge $e = (u, v)$ we say that $e$ is *directed from $u$ to $v$*.
- $G = (V, E)$ *undirected* if for all $v, w \in V$:

  $$(v, w) \in E \iff (w, v) \in E.$$

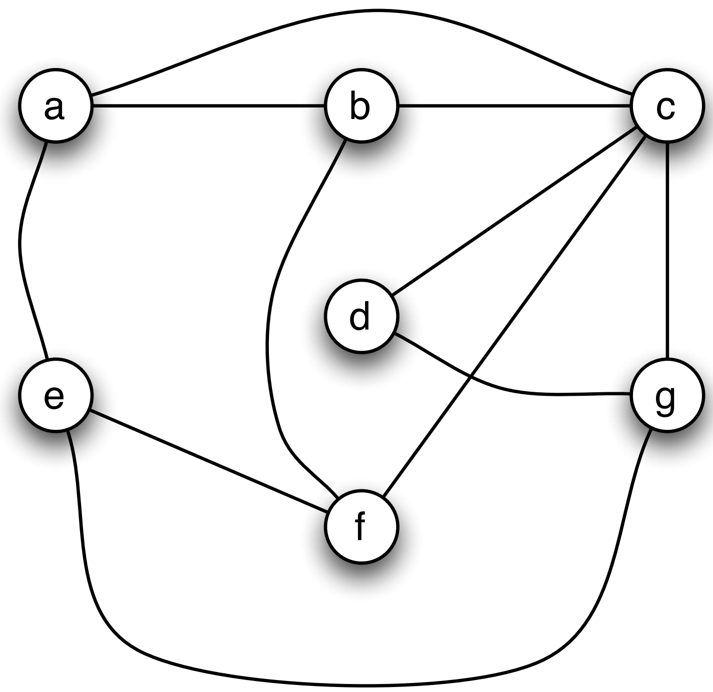
  Otherwise *directed*.
  Directed $\sim$ *arrows* (one-way)
  Undirected $\sim$ *lines* (two-way)
- We assume $V$ is finite, hence $E$ is also finite.

# A directed graph

$$\begin{aligned}
G &= (V, E),\\
V &= \{0, 1, 2, 3, 4, 5, 6\},\\
E &= \{(0,2), (0,4), (0,5), (1,0), (2,1), (2,5),\\
&\qquad (3,1), (3,6), (4,0), (4,5), (6,3), (6,5)\}.
\end{aligned}$$



# An undirected graph



## Examples

- *Road Maps.*
  Edges represent streets and vertices represent crossings (junctions).
- *Computer Networks*.
  Vertices represent computers and edges represent network connections (cables) between them.
- *The World Wide Web*.
  Vertices represent webpages, and edges represent hyperlinks.
- . . .

# Adjacency matrices

Let $G = (V, E)$ be a graph with $n$ vertices. Vertices of $G$ are numbered $0, \ldots, n-1$.

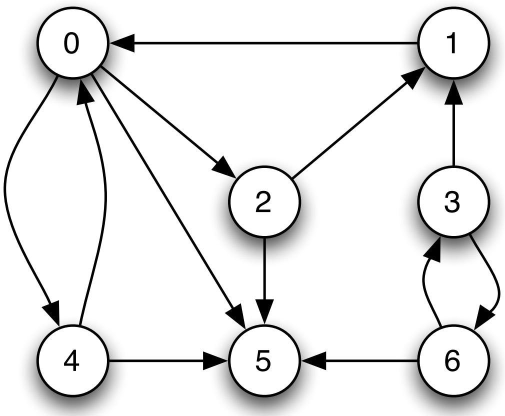
The *adjacency matrix* of $G$ is the $n \times n$ matrix

$$A = (a_{ij})_{0 \leq i,j \leq n-1}$$

with

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge from vertex } i \text{ to vertex } j; \\ 0, & \text{otherwise.} \end{cases}$$

# Adjacency matrix (Example)

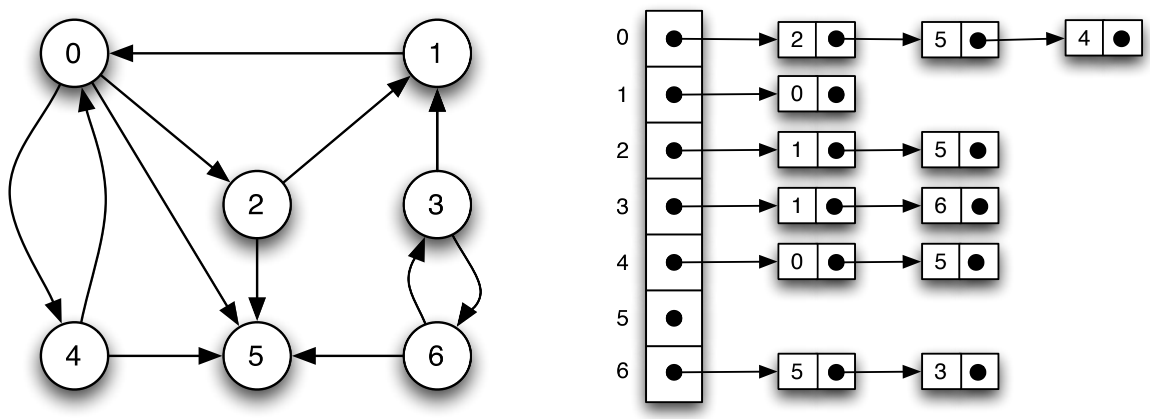



$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

# Adjacency lists

Array with one entry for each vertex $v$, which is a list of all vertices adjacent to $v$.

**Example**



# Quick Question

Given: graph $G = (V, E)$, with $n = |V|$, $m = |E|$.
For $v \in V$, we write $in(v)$ for in-degree, $out(v)$ for out-degree.

Which data structure has faster (asymptotic) worst-case running-time, for *checking if w is adjacent to v*, for a given pair of vertices?

1. Adjacency list is faster.
2. Adjacency matrix is faster.
3. Both have the same asymptotic worst-case running-time.
4. It depends.

Answer: 2. For an Adjacency Matrix we can check in $\Theta(1)$ time.
An adjacency list structure takes $\Theta(1 + out(v))$ time.

# Quick Question

Given: graph $G = (V, E)$, with $n = |V|$, $m = |E|$.
For $v \in V$, we write $in(v)$ for in-degree, $out(v)$ for out-degree.

Which data structure has faster (asymptotic) worst-case running-time, for *visiting all vertices w adjacent to v*, for a given vertex $v$?

1. Adjacency list is faster.
2. Adjacency matrix is faster.
3. Both have the same asymptotic worst-case running-time.
4. It depends.

Answer: 3. Adjacency matrix requires $\Theta(n)$ time always.
Adjacency list requires $\Theta(1 + out(v))$ time.
In worst-case $out(v) = \Theta(n)$.

# Adjacency Matrices vs Adjacency Lists

| | adjacency matrix | adjacency list |
|---|---|---|
| Space | $\Theta(n^2)$ | $\Theta(n+m)$ |
| Time to check if $w$ adjacent to $v$ | $\Theta(1)$ | $\Theta(1+out(v))$ |
| Time to visit all $w$ adjacent to $v$. | $\Theta(n)$ | $\Theta(1+out(v))$ |
| Time to visit all edges | $\Theta(n^2)$ | $\Theta(n+m)$ |

# Sparse and dense graphs

$G = (V, E)$ graph with $n$ vertices and $m$ edges

**Observation:** $m \leq n^2$

- $G$ *dense* if $m$ close to $n^2$
- $G$ *sparse* if $m$ much smaller than $n^2$

# Graph traversals

A *traversal* is a strategy for visiting all vertices of a graph while respecting edges.

*BFS* = *breadth-first search*

*DFS* = *depth-first search*

**General strategy:**

1. Let $v$ be an arbitrary vertex
2. Visit all vertices reachable from $v$
3. If there are vertices that have not been visited, let $v$ be such a vertex and go back to (2)

# Graph Searching (general Strategy)

**Algorithm** searchFromVertex($G$, $v$)

```
mark v
put v onto schedule S
while schedule S is not empty do
    remove a vertex v from S
    for all w adjacent to v do
        if w is not marked then
            mark w
            put w onto schedule S
```

**Algorithm** search($G$)

```
ensure that each vertex of G is not marked
initialise schedule S
for all v ∈ V do
    if v is not marked then
        searchFromVertex(G, v)
```

# Three colour view of vertices

- Previous algorithm has vertices in one of two states: *unmarked* and *marked*. Progression is

  $$\textit{unmarked} \longrightarrow \textit{marked}$$

- Can also think of them as being in one of three states (represented by colours):
  - *White*: not yet seen (not yet investigated).
  - *Grey*: put on schedule (under investigation).
  - *Black*: taken off schedule (completed).

  Progression is

  $$\textit{white} \longrightarrow \textit{grey} \longrightarrow \textit{black}$$

We will use the three colour scheme when studying an algorithm for topological sorting of graphs.

# BFS

Visit all vertices reachable from $v$ in the following order:

- $v$
- all neighbours of $v$
- all neighbours of neighbours of $v$ that have not been visited yet
- all neighbours of neighbours of neighbours of $v$ that have not been visited yet
- etc.

# BFS (using a Queue)

**Algorithm** bfs($G$)

1. Initialise Boolean array *visited*, setting all entries to FALSE.
2. Initialise *Queue* $Q$
3. **for all** $v \in V$ **do**
4.     **if** $visited[v] = \text{FALSE}$ **then**
5.         bfsFromVertex($G$, $v$)

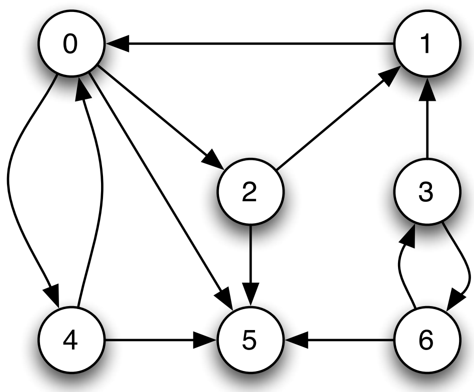
## BFS (using a Queue)

**Algorithm** bfsFromVertex($G$, $v$)

```
visited[v] = TRUE
Q.enqueue(v)
while not Q.isEmpty() do
    v ← Q.dequeue()
    for all w adjacent to v do
        if visited[w] = FALSE then
            visited[w] = TRUE
            Q.enqueue(w)
```

**Algorithm** bfsFromVertex($G$, $v$)

```
visited[v] = TRUE
Q.enqueue(v)
while not Q.isEmpty() do
        v ← Q.dequeue()
        for all w adjacent to v do
                if visited[w] = FALSE then
                        visited[w] = TRUE
                        Q.enqueue(w)
```



# Quick Question

Given a graph $G = (V, E)$ with $n = |V|$, $m = |E|$, what is the worst-case running time of BFS, in terms of $m, n$?

1. $\Theta(m + n)$
2. $\Theta(n^2)$
3. $\Theta(mn)$
4. Depends on the number of components.

Answer: 1. To see this need to be careful about bounding running time for the loop at lines 5–8.
*Must* use the Adjacency List structure.

Answer: 2. if we use adjacency matrix representation.

# DFS

Visit all vertices reachable from $v$ in the following order:

- $v$
- some neighbour $w$ of $v$ that has not been visited yet
- some neighbour $x$ of $w$ that has not been visited yet
- etc., until the current vertex has no neighbour that has not been visited yet
- Backtrack to the first vertex that has a yet unvisited neighbour $v'$.
- Continue with $v'$, a neighbour, a neighbour of the neighbour, etc., backtrack, etc.

## DFS (using a stack)

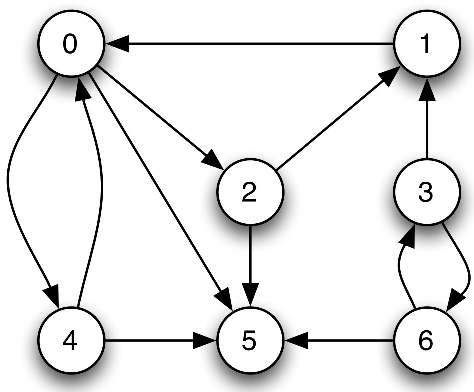
**Algorithm** dfs($G$)

1. Initialise Boolean array *visited*, setting all to FALSE
2. Initialise *Stack* $S$
3. **for all** $v \in V$ **do**
4.     **if** $visited[v] =$ FALSE **then**
5.         dfsFromVertex($G, v$)

# DFS (using a stack)

**Algorithm** dfsFromVertex($G$, $v$)

```
S.push(v)
while not S.isEmpty() do
    v ← S.pop()
    if visited[v] = FALSE then
        visited[v] = TRUE
        for all w adjacent to v do
            S.push(w)
```



# Recursive DFS

**Algorithm** dfs($G$)

1. Initialise Boolean array *visited*
   by setting all entries to FALSE
2. **for all** $v \in V$ **do**
3.     **if** $visited[v] = \text{FALSE}$ **then**
4.         dfsFromVertex($G$, $v$)

**Algorithm** dfsFromVertex($G$, $v$)

1. $visited[v] \leftarrow \text{TRUE}$
2. **for all** $w$ adjacent to $v$ **do**
3.     **if** $visited[w] = \text{FALSE}$ **then**
4.         dfsFromVertex($G$, $w$)

## Analysis of DFS

$G = (V, E)$ graph with $n$ vertices and $m$ edges

Without recursive calls:

- dfs($G$): time $\Theta(n)$
- dfsFromVertex($G, v$): time $\Theta(1 + \text{out-degree}(v))$

Overall time:

$$
\begin{aligned}
T(n, m) &= \Theta(n) + \textstyle\sum_{v \in V} \Theta(1 + \text{out-degree}(v)) \\
&= \Theta\Big(n + \textstyle\sum_{v \in V}(1 + \text{out-degree}(v))\Big) \\
&= \Theta\Big(n + n + \textstyle\sum_{v \in V} \text{out-degree}(v)\Big) \\
&= \Theta\Big(n + \textstyle\sum_{v \in V} \text{out-degree}(v)\Big) \\
&= \Theta(n + m)
\end{aligned}
$$