Introduction to Algorithms

1.1 Introduction

The Algorithms and Data Structures thread of Informatics 2B deals with the issues of how to store data efficiently and how to design efficient algorithms for basic problems such as sorting and searching. This thread is taught by Kyriakos Kalorkoti (KK).

There will generally be a lecture note for each lecture of this thread—these are adaptations of earlier notes of Don Sannella, John Longley, Martin Grohe, Mary Cryan and many others from the School of Informatics, University of Edinburgh. Some exercises will be included at the end of each lecture note, attempting these (in addition to the tutorial exercises) is a good way of reinforcing your understanding of the material.

Prerequisites. You have seen a few algorithms and data structures in Informatics 1B. Amongst these are sorting algorithms (MergeSort and Selection Sort); you should also understand the principle of recursion. Arrays, lists, and trees were also introduced in Inf1B. If for some reason you didn’t take Inf1B, then you should check the notes on these topics.

In addition to the Computer Science prerequisites, you will need to know some mathematics. We will assume that you are familiar with: proofs by induction, series and sums, recurrences, graphs and trees. Reminders of many of these will be included in the lectures.

Textbooks. The lecture notes provide the essentials for each topic, for further material or a different slant you should consult an appropriate book. Two recommended (though not required) algorithms textbooks for this thread are:


Both of these books cover most of the Algorithms and Data Structures material of Informatics 2B. [GT] puts a lot of emphasis on well-designed JAVA implementations of the algorithms, whereas [CLRS] is more theoretical. Of the two books, [GT] covers the course more closely (and is easier to read). However, [CLRS] is the textbook used in the third year Algorithms and Data Structures (ADS) course. Therefore if you expect to be taking this course (quite likely if you are on the CS degree) it is clearly a good idea to buy [CLRS], if you do feel the need to make a purchase at all. It is comprehensive and therefore makes an excellent reference book.
progress.

1.2 Evaluating Algorithms

Intuitively speaking, an algorithm is a step-by-step procedure (a “recipe”) for performing a task. A data structure is a systematic way of organizing data and making it accessible in certain ways. The two tasks of designing algorithms, and designing data structures, are closely linked; a good algorithm needs to use data structures that are suited to the problem. You have already seen some examples of data structures (arrays, linked lists, and trees) and some examples of algorithms (such as Linear search, Binary search, Selection Sort, and Mergesort) in Informatics 1B. Throughout this course we will cover these in more detail (a reminder of each of them will be included as well) and introduce more.

How do we decide whether an algorithm is good or not, or how good it is? This is the question we are going to to discuss in this introductory lecture.

The three most important criteria for an algorithm are correctness, efficiency, and simplicity. Correctness is clearly the most important criterion for evaluating an algorithm. If the algorithm is not doing what it is supposed to do, it is worthless. All the algorithms we present in this course are guaranteed to perform their tasks correctly, although we will not always cover the proofs. Some proofs of correctness are simple while other, more subtle, algorithms require complicated proofs of correctness.

The main focus of this thread will be on efficiency. We would like our algorithms to make efficient use of the computer’s resources. In particular, we would like them to run as fast as possible using as little memory as possible (though there is often a trade off here). To keep things simple, we will concentrate on the running time of the algorithms for a large part of this thread, and not worry too much about the amount of space (i.e., amount of memory) they use. However, in our lectures on algorithms for the internet, space will be an issue since these deal with huge quantities of data.

The third important criterion of an algorithm is simplicity. We would like our algorithms to be easy to understand and implement. Unfortunately, there is often (but not always) a trade-off between efficiency and simplicity: more efficient algorithms tend to be more complicated. Which of the two criteria is more important depends on the particular task. If a program is used only once or a few times, the cost of implementing a very efficient but complicated algorithm may exceed the cost of running a simple but inefficient one. On the other hand, if a program will be used very often, the cost of running an inefficient algorithm over and over again may exceed by far the cost of implementing an efficient but complicated one. It is up to the programmer (or system designer) to decide which way to go in any particular situation. In some application areas (e.g., real time control systems) we cannot afford inefficiency even for very rare events. For example the shutting down of some critical reaction in a chemical factory should be a rare event but it must be done in good time when necessary.

1.3 Measuring the Running Time

The running time of a program depends on a number of factors such as:

1. The running time of the algorithm on which the program is based.
2. The input given to the program.
3. The quality of the implementation and the quality of the code generated by the compiler.
4. The machine used to execute the program.

Warning. These are not the only factors. An approach that uses heavy recursion can often be slow when it is coded as a program (especially in an imperative language such as JAVA—functional languages such as Haskell are better at managing recursion efficiently). Moreover, JAVA (the language we use in Inf2B) has a heavy garbage collection overhead, so when we estimate running times with JAVA’s System.currentTimeMillis() command, the cost of garbage collection often drown out the true running time of the computation (unfortunately, for this reason, we often don’t see the effect of an efficient algorithm until the input is very large).

Anyhow, in developing a theory of efficiency, we don’t want a theory that is only valid for algorithms implemented by a particular programmer using a certain programming language as well as compiler and being executed on a particular machine. Therefore, in designing algorithms and data structures, we will abstract away from all these factors. We do this in the following way: We describe our algorithms in a pseudo-code that resembles programming languages such as JAVA or C, but is less formal (since it is only intended to be read and understood by humans). We assume that each instruction of our pseudo-code can be executed in a constant amount of time (more precisely, in a constant number of machine cycles that does not depend on the actual values of the variables involved in the statement), no matter which “real” programming language it is implemented in and which machine it is executed on. This can best be illustrated by a very simple example, Linear search, which you are very likely to have seen before.

Example 1.1. Algorithm 1.2 is the pseudo-code description of an algorithm that searches for an integer in an array of integers. Compare this with the actual implementation as a JAVA method displayed in Figure 1.3. The easiest way to implement most of the algorithms described in this thread is as static JAVA methods. We can embed them in simple test applications such as the one displayed in Figure 1.4.

The pseudocode is fairly self explanatory, note that we use indentation to indicate scope. This avoids unnecessary clutter and works well for our purposes because our pseudocode tends to be quite short.

We now want to get an estimate of the running time of linSearch when the input consists of the array \( A = \{2, 4, 3, 5, 6, 1, 0, 9, 7\} \) and the integer \( k = 0 \). The length of \( A \) is 10, and \( k \) occurs with index 7.

What we would like to estimate is the number of machine cycles an execution of the algorithm requires. More abstractly, instead of machine cycles we usually
Algorithm \texttt{linSearch}(A,k)  
\textbf{Input:} An integer array $A$, an integer $k$  
\textbf{Output:} The smallest index $i$ with $A[i] = k$, if such an $i$ exists, or $-1$ otherwise.  
1. \textbf{for} $i \leftarrow 0$ to $A.length - 1$ \textbf{do}  
2. \textbf{if} $A[i] = k$ \textbf{then}  
3. \textbf{return} $i$  
4. \textbf{return} $-1$  

\textbf{Algorithm 1.2} 

\begin{verbatim}
public static int linSearch(int[] A, int k) {
    for(int i = 0; i < A.length; i++)
        if ( A[i] == k )
            return i;
    return -1;
}
\end{verbatim}

\textbf{Figure 1.3.} An implementation of Algorithm 1.2 in JAVA

To compute the number of computation steps executed by Algorithm 1.2 on input $A = \{2, 4, 3, 5, 6, 1, 0, 9, 7\}$ and $k = 10$ we have to find out how often each line is executed. Line 1 is executed 8 times (for the values $i = 0, \ldots , 7$). Line 2 is also executed 8 times. Line 3 is executed once, and Line 4 is never executed. Thus overall, for the particular input $A$ and $k$, at most

$$8c_1 + 8c_2 + c_3$$

computation steps are performed. A similar analysis shows that on input $A = \{2, 4, 3, 5, 6, 1, 0, 9, 7\}$ and $k = 10$ at most

$$11c_1 + 10c_2 + c_4$$

computation steps are performed. (Line 1 is performed 11 times, the last time to check that $11 > 10 = A.length$.)

So far, this looks like a tedious exercise. All we have obtained is a dubious-looking estimate, involving unknown constants, on the number of computation steps performed by our algorithm on two particular inputs. We would like to have an estimate on the number of computation steps performed on arbitrary inputs. But clearly, we cannot just disregard the input: an execution of our algorithm can be expected to perform more computation steps if the input array has length 100,000 than if the input array has length 10. Therefore, we define the \textbf{running time} of the algorithm as a function of the size of the input, which in Example 1.1 is the length of $A$ (plus 1, for the value of $k$). But even for input arrays of the same length the execution time may vary greatly, depending on where the key $k$ actually appears in the array. We resolve this by giving an estimate on the number of computation steps needed in the \textbf{worst case}.

In these notes the symbol $\mathbb{N}$ denotes the set \{0, 1, 2, 3, \ldots \} of natural numbers including 0.\footnote{Some authors exclude 0 from the natural numbers (which I find rather unnatural)—KK.}
by A on an input of size n.

Remark 1.6. Note that for a given size there are only finitely many possible inputs of that size. For example if we are considering arrays of size n each location can only hold integers up to some maximum size. Let’s say there are B of these, then the number of arrays of size n is $B^n$ [why?].

The point of this observation is that $T_n$ is always well defined: for a given n we just have a finite number of runtimes to consider so there will indeed be a maximum. If there were infinitely many possible runtimes there might not be a maximum since the runtime numbers could keep growing.

Remark 1.7. For most of our algorithms we will represent the size of the input by one quantity (usually denoted n). In some cases is is more convenient to use two or more such quantities, each corresponding to the size of an input parameter (e.g., two arrays). The definition above has an obvious counterpart for this situation.

Remark 1.8. There is also a notion of average time complexity (it is not central to Inf2B). Average-case complexity is measured by considering all possible sets of inputs for a given problem, and taking the average of the runtimes for all those inputs. [GT] and [CLRS] have more details of average case complexity.

Example 1.1 (cont’d). Let us analyse the worst-case time complexity of algorithm linSearch. It is obvious that the worst-case inputs are those where either $i$ only appears as the last entry of $A$ or where $k$ does not appear at all (this is obvious because in these cases, we have to execute the loop $A.length$ times). For more complicated situations the worst case is not necessarily so obvious.

Suppose $A$ has length n. If $k$ appears at the last entry, the number of computation steps performed is at most

$$c_1n + c_2n + c_3 = (c_1 + c_2)n + c_3,$$

and if $k$ does not appear at all it is

$$c_1(n + 1) + c_2n + c_3 = (c_1 + c_2)n + (c_1 + c_4).$$

Thus the time complexity of linSearch is

$$T_{\text{linSearch}}(n) \leq \max \{ (c_1 + c_2)n + c_3, (c_1 + c_2)n + (c_1 + c_4) \}
= (c_1 + c_2)n + \max \{c_3, c_4\}$$

Since we do not know the constants $c_1, c_2, c_3, c_4$, we cannot tell which of the two values is larger. But the important thing we can see from our analysis is that the time complexity of linSearch is a function of the form

$$an + b,$$

for some constants $a, b$. This is a linear function, therefore we say that linSearch requires linear time.

We can compare linSearch with another algorithm, which you are very likely to have seen before:

Example 1.9. Algorithm 1.10 is the pseudo-code of an algorithm that searches for an integer in a sorted array of integers using binary search. Note that binary search only works correctly on sorted input arrays [why?]. In this case we assume the input array is sorted in increasing order, there is an obvious modification for arrays that are sorted in decreasing order.

Let us analyse the algorithm. Again, we assume that each execution of a line of code only requires constant time\(^3\) and we let $c_1$ be the time required to execute Line 1 once. However, for the recursive calls in Lines 8 and 10, $c_8$ and $c_{10}$ only account for the number of computation steps needed to initialise the recursive call (i.e., to write the local variables on the stack etc.), but not for the number of steps needed to actually execute binarySearch($A, k, i_2, j_2$) and binarySearch($A, k, j_1 + 1, i_2$).

The size of an input $A, k, i_1, i_2$ of binarySearch is

$$n = i_2 - i_1 + 1,$$

i.e., the number of entries of the array $A$ to be investigated. If we search the whole array, i.e., if $i_1 = 0$ and $i_2 = A.length - 1$ then $n$ is just the number of entries of $A$ as before. Then we can estimate the time complexity of binarySearch as follows:

$$T_{\text{binarySearch}}(n) \leq \sum_{i=1}^{n} c_8 + T_{\text{binarySearch}}([n/2]).$$

Here we add up the number of computation steps required to execute all lines of code and the time required by the recursive calls. Note that the algorithm makes

\(^3\)Note that this means we are assuming that compound structures, such as arrays, are passed by reference. If local copies were made there would be a very significant effect on runtime, usually resulting in a very inefficient algorithm. In this example setting up each recursive call would cost time proportional to the size of the array being passed rather than constant time.
at most one recursive call, either in Line 8 or in Line 10. Observing that
\[(j - 1) - i_1 + 1 = \left\lfloor \frac{i_2 + i_2}{2} \right\rfloor - i_1 = \left\lfloor \frac{i_2 - i_1}{2} \right\rfloor = \left\lfloor \frac{n - 1}{2} \right\rfloor \leq \lfloor n/2 \rfloor\]
and
\[i_2 - (j - 1) + 1 = i_2 - \left\lfloor \frac{i_2 + i_2}{2} \right\rfloor = \left\lfloor \frac{i_2 - i_1}{2} \right\rfloor = \left\lfloor \frac{n - 1}{2} \right\rfloor \leq \lfloor n/2 \rfloor,
we see that this recursive call is made on an input of size at most \(\lfloor n/2 \rfloor\). Thus the cost of the recursive call is at most \(T_{\text{binarySearch}}(\lfloor n/2 \rfloor)\). Note that we have assumed that \(T_{\text{binarySearch}}\) is a non-decreasing function of its argument. This is clearly a very plausible assumption and is easy to establish (we will not do so here in order not to disrupt the key point). We let \(c = \sum_{i \geq 1} c_i\) and claim that
\[T_{\text{binarySearch}}(n) \leq c(\log(n) + 2)\]  
(1.2)
for all \(n \in \mathbb{N}\) with \(n \geq 1\) (recall that \(\log(n) = \log_2(n)\), i.e., “log to the base 2”).

We prove (1.2) by induction on \(n\). For the induction base \(n = 1\) we obtain
\[T_{\text{binarySearch}}(1) \leq c + T_{\text{binarySearch}}(0) \leq 2c = c(\log(1) + 2).

The inequality \((*)\) holds because \(T_{\text{binarySearch}}(0) \leq c_1 + c_2 \leq c\).

For the inductive step, let \(n \geq 2\). We assume that we have already proved (1.2) for all \(n' < n\) and now want to prove it for \(n\). We have
\[T_{\text{binarySearch}}(n) \leq c + T_{\text{binarySearch}}(\lfloor n/2 \rfloor) \leq c + c(\log(\lfloor n/2 \rfloor) + 2) \leq c + c(\log(n) - 1 + 2) \quad \text{(since \(\log(n) \geq \log(n/2) = \log(n) - 1\))}
= c(\log(n) + 2).

This proves (1.2).

Remark 1.11. The preceding proof by induction gives us an unimpeachable demonstration that (1.2) is indeed correct. However it provides no intuitive insight as to why. Let us reconsider the problem. What we know is that when \(\text{binarySearch}\) is called on an array of size \(n\) it does a constant amount of work (which we have denoted by \(c\)) and then recurses on an array of size around \(n/2\) (unless it has finished). Well at this level it also does a constant amount of work (i.e., \(c\)) and then recurses on an array of size around \(n/2^2\) (unless it has finished). So in the worst case it keeps recursing having done \(c\) amount of work each time. When does this stop? It does so when we get to an array of size 1 and then it makes one more call (we could avoid this by changing the code slightly). After \(r\) recursive calls the size of the array is around \(n/2^r\) so we get to a size 1 array when \(2^r = n\), i.e., when \(r = \log(n)\) (strictly we should adjust this as \(r\) must be an integer, but we are just trying to get a rough idea of the situation). Thus the number of recursive calls is at most \(\log(n) + 1\) and each of them does \(c\) amount of work but we also do \(c\) amount of work before the first recursive call. This gives us \(c(\log(n) + 2)\) as an upper bound for the total amount of work (or time). Having obtained this we can then proceed to prove its correctness by induction.

Hence we get the following bound on the running time of \(\text{inSearch}\):
\[T_{\text{inSearch}}(n) \leq an + b\]
for suitable constants \(a, b\), and for the running time of \(\text{binarySearch}\) we have
\[T_{\text{binarySearch}}(n) \leq c(\log(n) + d)\]
for suitable constants \(c, d\). Now what does this mean? After all, we don’t know the constants \(a, b, c, d\). Well, even if we don’t know the exact constants, it tells us that binary search is much faster than linear search. Remember where the constants came from: they were upper bounds on the number of machine cycles needed to execute a few very basic instructions. So they won’t be very large. To support our claim that binary search is faster than linear search, let us make an assumption on the constants that strongly favours sequential search: Assume that \(a = b = 10\) and \(c = d = 1000\). Table 1.12 shows the number of computation steps performed by the two algorithms for increasing input sizes. Of course for small input arrays, the large constants we assumed for \(\text{binarySearch}\) have a strong impact. But already for arrays of size 10,000, binarySearch is more than 6 times faster, and for arrays of size 1,000,000, binarySearch is about 50 times faster. Figure 1.13 shows the graph of the two functions up to \(n = 10000\) (going much higher tends to conceal the crossover point).

The message of Table 1.12 is that the constant factors \(a, b, c, d\) are dominated by the relationship between the growth rate of \(n\) versus \(\log(n)\). So what will be most important in analysing the efficiency of an algorithm will be working out its underlying growth rate, forgetting constants.

It is easy to understand the behaviour of the two functions. By definition
\[n = 2^w(n)\]
Thus in order to increase the value of \(\log(n)\) by 1 we must double the value of \(n\). So to increase \(\log(n)\) by 10 we must multiply the value of \(n\) by \(2^{10} = 1024\) and to increase the value of \(\log(n)\) by 20 we must multiply the value of \(n\) by \(2^{20} = 1.048,576\). This behaviour is an example of exponential growth.
We have encoded both of the `linSearch` and `binarySearch` algorithms in JAVA, and have run tests on arrays of various sizes (performed on a DICE machine in January 2008), in order to see how the details of Table 1.12 influence the performance of the algorithms in practice. A line of the table is to be read as follows. Column 1 is the input size. Each of the two algorithms was run on 200 randomly generated arrays of this size. Column 2 is the worst case time (over the 200 runs) of `linSearch`; Column 3 the average time. Similarly, Column 4 is the worst-case time of `binarySearch` and Column 5 the average time.

<table>
<thead>
<tr>
<th>size</th>
<th>wc linS</th>
<th>avc linS</th>
<th>wc binS</th>
<th>avc binS</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>100</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>1000</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>10000</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>200000</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>400000</td>
<td>3 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>600000</td>
<td>3 ms</td>
<td>1.3 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>800000</td>
<td>3 ms</td>
<td>1.5 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>1000000</td>
<td>5 ms</td>
<td>2.1 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>2000000</td>
<td>7 ms</td>
<td>3.7 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>4000000</td>
<td>12 ms</td>
<td>6.9 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>6000000</td>
<td>24 ms</td>
<td>11.6 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
<tr>
<td>8000000</td>
<td>24 ms</td>
<td>15.6 ms</td>
<td>≤ 1 ms</td>
<td>≤ 1 ms</td>
</tr>
</tbody>
</table>

Table 1.14. Running Times of `linSearch` and `binarySearch`

The complete test application we used is `Search.java`, and can be found on the Inf2B webpages. The way the running time is measured here is quite primitive: The current machine time\(^4\) before starting the algorithm is subtracted from the time after starting the algorithm. This is not a very exact way of measuring the running time in a multi-tasking operating system. However, in current JAVA compilers, there is unfortunately no way of turning off the garbage collection facility, so a rough estimate is the best we can hope for.

Before we leave this comparison it is well worth thinking about the reason for the far superior efficiency of `binarySearch` over `linSearch`. Compare the following observations on what happens after one comparison:

- `linSearch`: we either succeed or have dismissed just one item of current data and the rest of the data to be examined.
- `binarySearch`: we either succeed or have dismissed around half of the current data with the other half remaining to be examined.

Clearly for both algorithms the worst case is that we do not succeed early. Now `linSearch` plods its way dismissing one item at a time while `binarySearch` dismisses half the data (and does so without doing any more work than `linSearch`)

\(^4\)Given by `java.lang.System.currentTimeMillis()`.
when dismissing the current item under consideration). An intuitive understanding such as this underpins the design of new efficient algorithms.

The preceding discussion shows that the replacement of the factor $n$ in the worst case runtime of $\text{linSearch}$ with $\lg n$ in the worst case runtime of $\text{binarySearch}$ is a huge improvement. We will see similar gains, e.g., when we consider sorting where various simple algorithms have worst case runtime proportional to $n^2$ whereas more sophisticated ones have worst case runtime proportional to $n \log n$. No improvement of hardware speed can match this, since such an improvement is by some constant factor.

### 1.4 The need for efficient algorithms

Efficiency is not usually a major issue for small problem sizes but in many applications the size of the data is immense. Figure 1.15 illustrates the problem very clearly.

With the impressive advances in hardware it might be tempting to assume that this will resolve the issue of efficiency. Figure 1.16 shows that this approach cannot work; the 1990 date has been left because it underlines the fact that the point being made is independent of the dates involved. Note, in particular the drastic effect that exponential runtimes have (in our table these are $2^n$ and $n!$). Of course we aim for the best of both worlds: efficient algorithms and fast hardware.

### 1.5 The need for theoretical analysis

Finally in this note we discuss why a theoretical analysis of algorithms is essential. It might at first seem to be sufficient to implement algorithms of interest to us and run some timing experiments. There are many problems with taking this approach exclusively; we will focus on the two most critical ones.

First of all our experiments can only go up to some upper bound on the size $n$ of the input. Thus they cannot show us the trend as $n$ increases. However we could just note here that in any realistic context there will indeed be an upper bound to $n$ albeit we might be hard put to it fix it precisely. Let’s concede this point, even though it is not as clear as it might seem.

The real problem is as follows. Consider one of the most fundamental computational problems: sorting objects on which we have a known linear order, let’s say integers. It is surely reasonable to expect that in many applications we will have $n = 100$ or indeed much more (i.e., the algorithm could be expected to sort 100 objects). Let’s consider trying to obtain a reliable estimate for the runtime of an algorithm just on this size of input (to keep things simple we will assume that each input consists of the integers $1, 2, \ldots, 100$ in some order). We will be generous and accept the timings based on just 1% of all possible inputs (of course this is very unreliable for worst case runtimes). How many experiments does this imply? There are $n!$ different ways of ordering $n$ distinct objects so this is the number of possible inputs. Thus we must carry out $n!/100$ experiments, i.e.,

$$n!$$

Note however that an inefficient algorithm that is run a huge number of times is a problem even if each instance is small. Furthermore, if we are dealing with safety critical systems then even the slightest delay can be a real problem.

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**Figure 1.15.** Time required to solve problems of size $n$ with an algorithm of the given runtime (algorithm runtime in $\mu s$, i.e., $10^{-6}$ seconds)

<table>
<thead>
<tr>
<th>Time</th>
<th>$n = 10$</th>
<th>$n = 20$</th>
<th>$n = 50$</th>
<th>$n = 100$</th>
<th>$n = 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \lg n$</td>
<td>33 $\mu$s</td>
<td>86 $\mu$s</td>
<td>282 $\mu$s</td>
<td>664 $\mu$s</td>
<td>10 ms</td>
</tr>
<tr>
<td>$n^2$</td>
<td>100 $\mu$s</td>
<td>400 $\mu$s</td>
<td>3 ms</td>
<td>10 ms</td>
<td>1 s</td>
</tr>
<tr>
<td>$n^3$</td>
<td>100 ms</td>
<td>3 s</td>
<td>5 mins</td>
<td>3 hrs</td>
<td>32 yrs</td>
</tr>
<tr>
<td>$2^n$</td>
<td>1 ms</td>
<td>1 s</td>
<td>36 yrs</td>
<td>$4 \times 10^{10}$ yrs</td>
<td></td>
</tr>
<tr>
<td>$n!$</td>
<td>4 s</td>
<td>774 cents</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 1.16.** Largest problem sizes solvable in 1 minute

<table>
<thead>
<tr>
<th>Fastest comp. in 1990</th>
<th>Comp. 1000 times faster</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \lg n$</td>
<td>1.5 trillion</td>
</tr>
<tr>
<td>$n^2$</td>
<td>8 million</td>
</tr>
<tr>
<td>$n^3$</td>
<td>570</td>
</tr>
<tr>
<td>$2^n$</td>
<td>46</td>
</tr>
<tr>
<td>$n!$</td>
<td>16</td>
</tr>
</tbody>
</table>

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Such a sample size might well provide a reliable estimate but we cannot know this without further reasoning about our algorithm. Here we are addressing the shortcomings of an approach based solely on experiments.
99! experiments. A direct calculation shows that
\[ 99! = 933262154439441526816992388562667004907198624656477913 \]
Now even if the algorithm could sort $10^{50}$ instances per second(!) the experiment would take at least
\[ \frac{99!}{60 \times 24 \times 3600 \times 10^{50}} \approx 2.951269209 \times 10^{98} \]
years to complete (we have been generous again and taken each year to be a leap year). Time enough for a cup of tea then!
This section must not be taken to imply that experiments have no value. However it is always essential to have a clear idea about the significance of any conclusions reached. When coupled with a good analysis experiments are an invaluable addition to our knowledge.

1.6 Reading Material
[CLRS], Chapter 1 and Sections 2.1-2.2.

Exercises
1. Consider the algorithm findMax (Algorithm 1.17) that finds the maximum entry in an integer array.

Algorithm findMax(A)

Input: An integer array A
Output: The maximum entry of A.
1. \( m \leftarrow A[0] \)
2. for \( i \leftarrow 1 \) to \( A.n\text{length} - 1 \) do
3. if \( A[i] > m \) then
4. \( m \leftarrow A[i] \)
5. return \( m \)

Algorithm 1.17

Show that there are constants \( c_1, d_1, c_2, d_2 \in \mathbb{N} \) such that
\[ c_1 n + d_1 \leq T_{\text{findMax}}(n) \leq c_2 n + d_2 \]
for all \( n \), where \( n \) is the length of the input array.
Argue that for every algorithm \( A \) for finding the maximum entry of an integer array of length \( n \) it holds that
\[ T_A(n) \geq n. \]