Constructions on Finite Automata

Informatics 2A: Lecture 4

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Determinization

- The subset construction

Closure properties of regular languages

- Union
- Intersection
- Complement

DFA minimization

- The problem
- An algorithm for minimization

Recap of Lecture 3

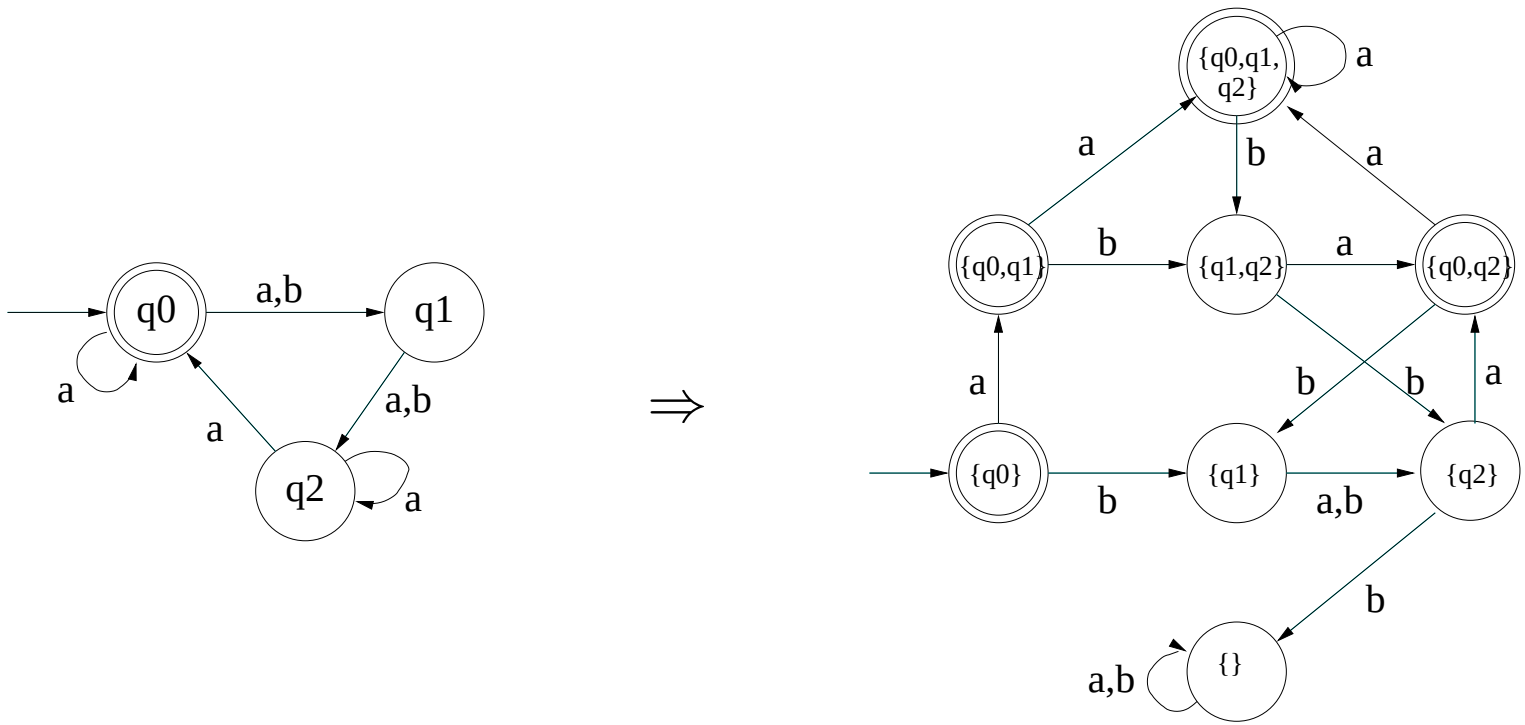
- A language is a set of strings over an alphabet Σ.
- A language is called regular if it is recognised by some NFA.
- DFAs are an important subclass of NFAs.
- (hinted) An NFA with n states can be determinized to an equivalent DFA with 2^n states, using the subset construction.
- Therefore the regular languages are exactly the languages recognised by DFAs.

The key insight

- The process we described on Friday for our example (what is the set of states we can reach after reading the next character) is a completely deterministic process! Given any current set of 'coloured' states, and any input symbol in Σ, there's only one right answer to the question: 'What will the new set of coloured states be?'
- What's more, it's a finite state process. A 'state' is simply a choice of 'coloured' states in the original NFA N.
  If N has n states, there are 2^n such choices.
- This suggests how an NFA with n states can be converted into a equivalent DFA with 2^n states.

The subset construction: example

Our 3-state NFA gives rise to a DFA with $2^3 = 8$ states. The states of this DFA are subsets of $\{q0, q1, q2\}$.



The accepting states of this DFA are exactly those that *contain* an accepting state of the original NFA.

The subset construction in general

Given an NFA $N = (Q, \Delta, S, F)$, we can define an equivalent DFA $M = (Q', \delta', s', F')$ (over the same alphabet Σ) like this:

- Q' is 2^Q, the set of all subsets of Q. (Also written $\mathcal{P}(Q)$.)
- $\delta'(A, u) = \{q' \in Q \mid \exists q \in A.\ (q, u, q') \in \Delta\}$. (Set of all states reachable via u from *some* state in A.)
- $s' = S$.
- $F' = \{A \subseteq Q \mid \exists q \in A.\ q \in F\}$.

It's then not hard to prove mathematically that $\mathcal{L}(M) = \mathcal{L}(N)$. (See Kozen for details.)

This process is called determinization.

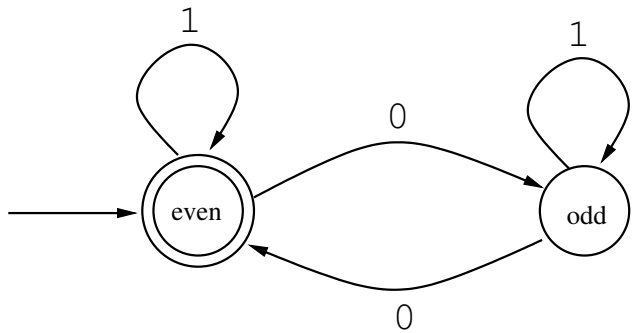
Coming up in lecture 6: Application of this process to efficient string searching.

Summary

- We've shown that for any NFA N, we can construct a DFA M with the same associated language.
- Since every DFA is also an NFA, the classes of languages recognised by DFAs and by NFAs coincide — these are the regular languages.
- Often a language can be specified more concisely by an NFA than by a DFA.
- We can automatically convert an NFA to a DFA, at the risk of an exponential blow-up in the number of states.
- To determine whether a string x is accepted by an NFA we do not need to construct the entire DFA, but instead we efficiently simulate the execution of the DFA on x on a step-by-step basis. (This is called just-in-time simulation.)

Question 1

Let M be the DFA shown on Friday:



Give a simple, concise definition of the strings that are in $\mathcal{L}(M)$.

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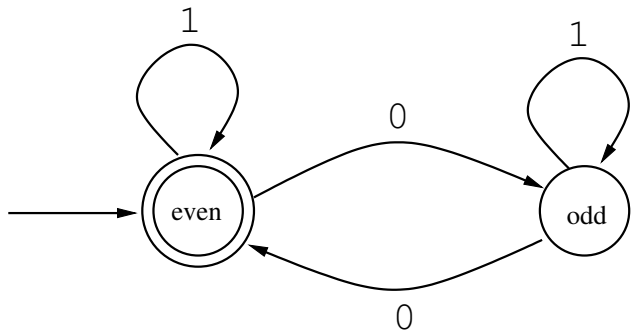

Give a simple, concise definition of the strings that are in $\mathcal{L}(M)$.

Answer: They are the strings containing an even number of 0's.

Question 2

Which of these three languages do you think are regular?

$$\begin{aligned} L_1 &= \{a, aa, ab, abbc\} \\ L_2 &= \{axb \mid x \in \Sigma^*\} \\ L_3 &= \{a^n b^n \mid n \geq 0\} \end{aligned}$$

If not regular, can you explain why not??

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Answers: L_1 is regular (easy to see that any finite set of strings is a regular language).

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Answers: L_1 is regular (easy to see that any finite set of strings is a regular language). L_2 is regular (easy to construct a DFA).

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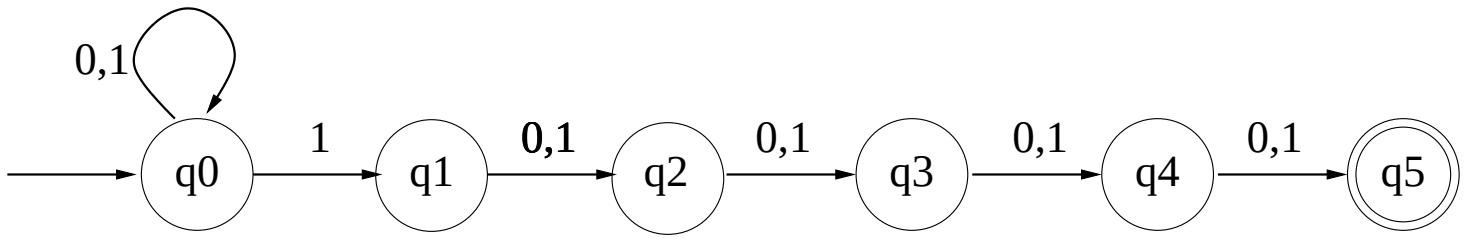
If not regular, can you explain why not??

Answers: L_1 is regular (easy to see that any finite set of strings is a regular language). L_2 is regular (easy to construct a DFA). L_3 is not regular. We shall see why in lecture 8.

Question 3

Consider our first example NFA over $\{0, 1\}$ (from Friday):



What is the number of states of the smallest DFA that recognises the same language?

Answer will be given in Lecture 5.

Union of regular languages

Consider the following little theorem:

If L_1 and L_2 are regular languages over Σ, so is $L_1 \cup L_2$.

This is dead easy to prove using NFAs.

Suppose $N_1 = (Q_1, \Delta_1, S_1, F_1)$ is an NFA for L_1, and $N_2 = (Q_2, \Delta_2, S_2, F_2)$ is an NFA for L_2.

We may assume $Q_1 \cap Q_2 = \emptyset$ (just relabel states if not).

Now consider the NFA

$$(Q_1 \cup Q_2,\ \Delta_1 \cup \Delta_2,\ S_1 \cup S_2,\ F_1 \cup F_2)$$

This is just N_1 and N_2 'side by side'. Clearly, this NFA recognizes precisely $L_1 \cup L_2$.

Number of states $= |Q_1| + |Q_2|$ — no state explosion!

Intersection of regular languages

If L_1 and L_2 are regular languages over Σ, so is $L_1 \cap L_2$.

Suppose $N_1 = (Q_1, \Delta_1, S_1, F_1)$ is an NFA for L_1, and $N_2 = (Q_2, \Delta_2, S_2, F_2)$ is an NFA for L_2.

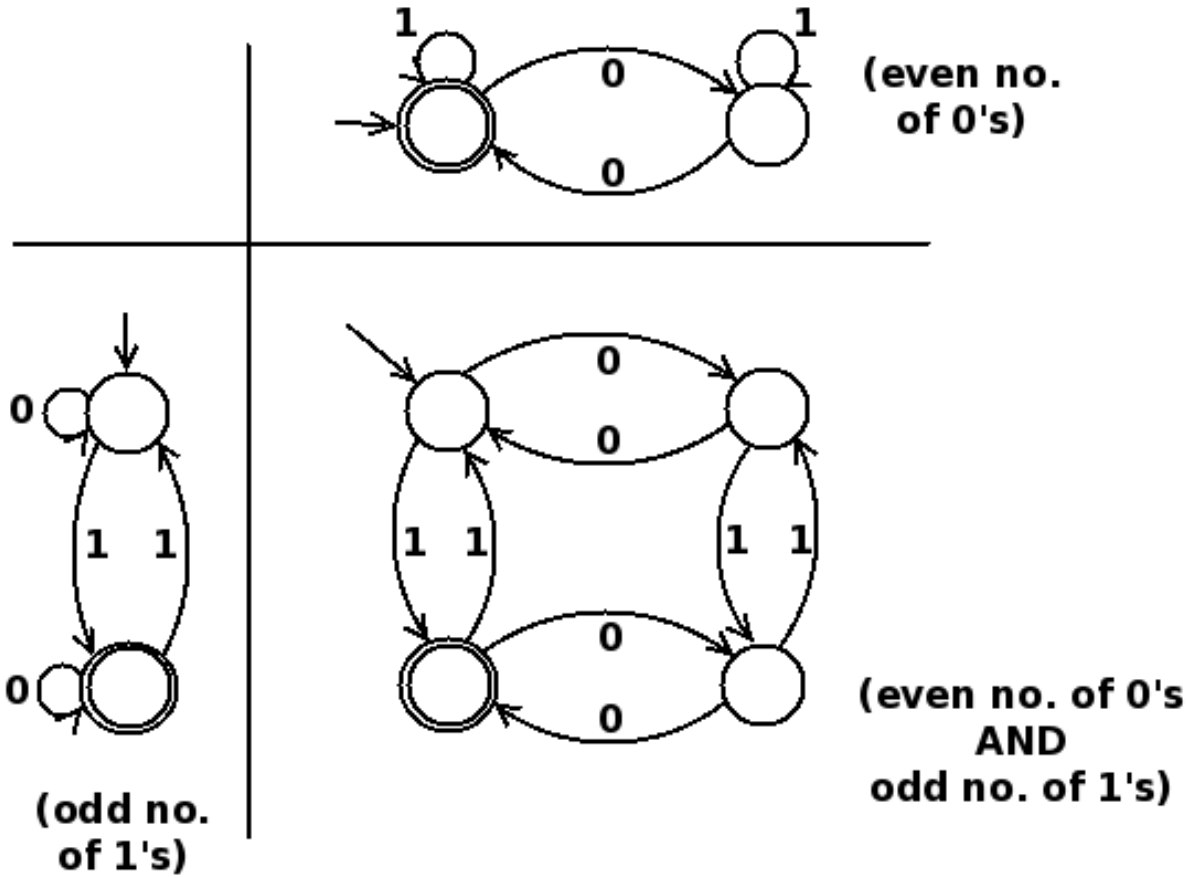
We define a product NFA $(Q',\ \Delta',\ S',\ F')$ by:

$$\begin{aligned} Q' &= Q_1 \times Q_2 \\ (q, r) \xrightarrow{a} (q', r') \in \Delta' &\iff q \xrightarrow{a} q' \in \Delta_1 \text{ and } r \xrightarrow{a} r' \in \Delta_2 \\ S' &= S_1 \times S_2 \\ F' &= F_1 \times F_2 \end{aligned}$$

Number of states $= |Q_1| \times |Q_2|$ — a bit more costly than union!

If N_1 and N_2 are DFAs then the product automaton is a DFA too.

Example of language intersection



Complement of a regular language

(Recall the set-difference operation,

$$A - B \;=\; \{x \in A \mid x \notin B\}$$

where A, B are sets.)

If L is a regular language over Σ, then so is $\Sigma^* - L$.

Suppose $N = (Q, \delta, s, F)$ is a DFA for L.

Then $(Q, \delta, s, Q - F)$ is a DFA for $\Sigma^* - L$. (We simply swap the accepting and rejecting states in N.)

Number of states $= |Q|$ — no blow up at all, but we are required to start with a DFA. This in itself has size implications.

The complement construction does not work if N is not deterministic!

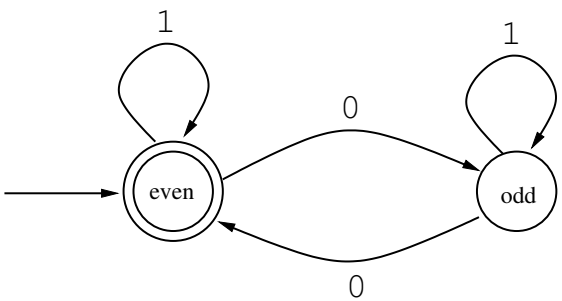
Closure properties of regular languages

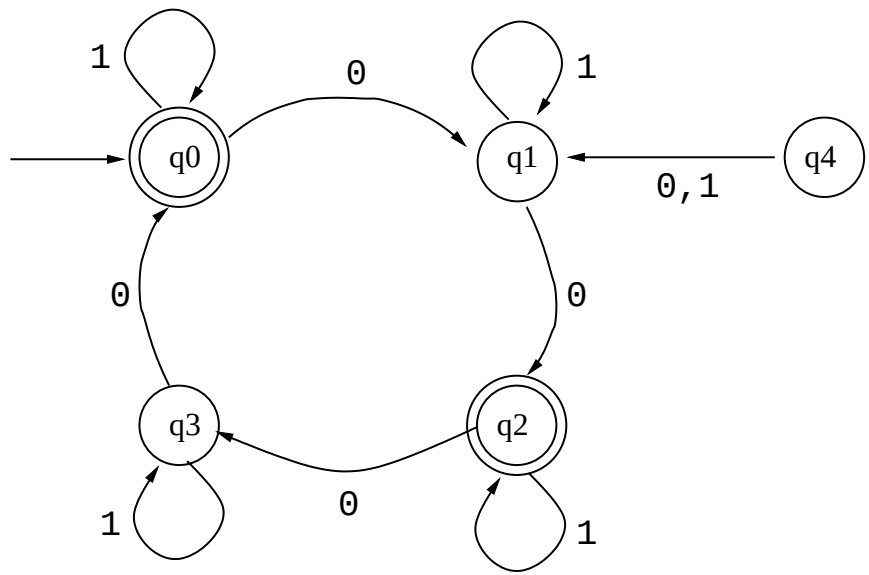
- We've seen that if both L_1 and L_2 are regular languages, then so are:
  - $L_1 \cup L_2$ (union)
  - $L_1 \cap L_2$ (intersection)
  - $\Sigma^* - L_1$ (complement)
- We sometimes express this by saying that regular languages are closed under the operations of union, intersection and complementation. ('Closed' used here in the sense of 'self-contained'.)
- Each closure property corresponds to an explicit construction on finite automata. Sometimes this uses NFAs (union), sometimes DFAs (complement), and sometimes the construction works equally well for both NFAs and DFAs (intersection).

The Minimization Problem

Determinization involves an exponential blow-up in the automaton. Is it sometimes possible to reduce the size of the resulting DFA?

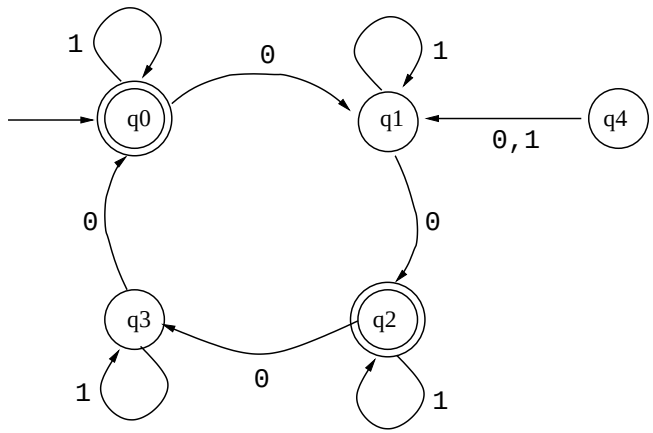
Many different DFAs can give rise to the same language, e.g.:





We shall see that there is always a unique smallest DFA for a given regular language.

DFA minimization



We perform the following steps to 'reduce' M above:

- Throw away unreachable states (in this case, q4).
- Squish together equivalent states, i.e. states q, q' such that: every string accepted starting from q is accepted starting from q', and vice versa. (In this case, q0 and q2 are equivalent, as are q1 and q3.)

Let's write $\text{Min}(M)$ for the resulting reduced DFA. In this case, $\text{Min}(M)$ is essentially the two-state machine on the previous slide.

Properties of minimization

The minimization operation on DFAs enjoys the following properties which characterise the construction:

- $\mathcal{L}(\text{Min}(M)) = \mathcal{L}(M)$.
- If $\mathcal{L}(M') = \mathcal{L}(M)$ and $|M'| \leq |\text{Min}(M)|$ then $M' \cong \text{Min}(M)$.

Here $|M|$ is the number of states of the DFA M, and $\cong$ means the two DFAs are isomorphic: that is, identical apart from a possible renaming of states.

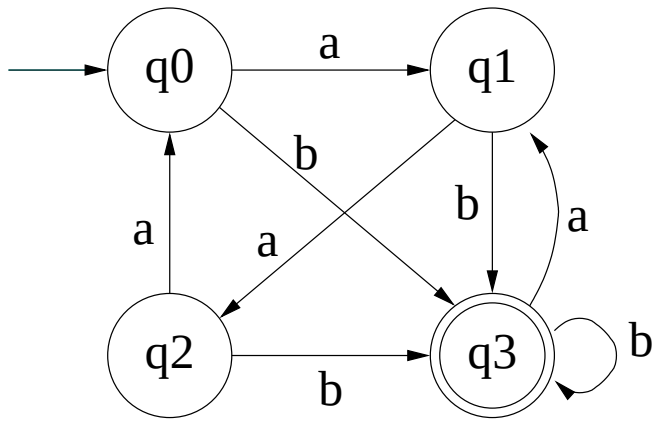
Two consequences of the above are:

- $\text{Min}(M) \cong \text{Min}(M')$ if and only if $\mathcal{L}(M) = \mathcal{L}(M')$.
- $\text{Min}(\text{Min}(M)) \cong \text{Min}(M)$.

For a formal treatment of minimization, see Kozen chapters 13–16.

Challenge question

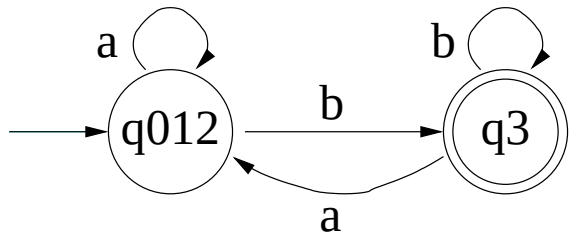
Consider the following DFA over $\{a, b\}$.



How many states does the minimized DFA have?

Solution

The minimized DFA has just 2 states:



The minimized DFA has been obtained by squishing together states q0, q1 and q2. Clearly q3 must be kept distinct.

Note that the corresponding language consists of all strings ending with b.

Minimization in practice

Let's look again at our definition of equivalent states:

> *states q, q' such that: every string accepted starting from q is accepted starting from q', and vice versa.*

This is fine as an abstract mathematical definition of equivalence, but it doesn't seem to give us a way to compute which states are equivalent: we'd have to 'check' infinitely many strings $x \in \Sigma^*$.

Fortunately, there's an actual algorithm for DFA minimization that works in reasonable time.

This is useful in practice: we can specify our DFA in the most convenient way without worrying about its size, then minimize to a more 'compact' DFA to be implemented e.g. in hardware.

An algorithm for minimization

First eliminate any unreachable states (easy).

Then create a table of all possible pairs of states (p, q), initially unmarked. (E.g. a two-dimensional array of booleans, initially set to false.) We mark pairs (p, q) as and when we discover that p and q cannot be equivalent.

1. Start by marking all pairs (p, q) where $p \in F$ and $q \notin F$, or vice versa.
2. Look for unmarked pairs (p, q) such that for some $u \in \Sigma$, the pair $(\delta(p, u), \delta(q, u))$ is marked. Then mark (p, q).
3. Repeat step 2 until no such unmarked pairs remain.

If (p, q) is still unmarked, can collapse p, q to a single state.

Illustration of minimization algorithm

Consider the following DFA over $\{a, b\}$.

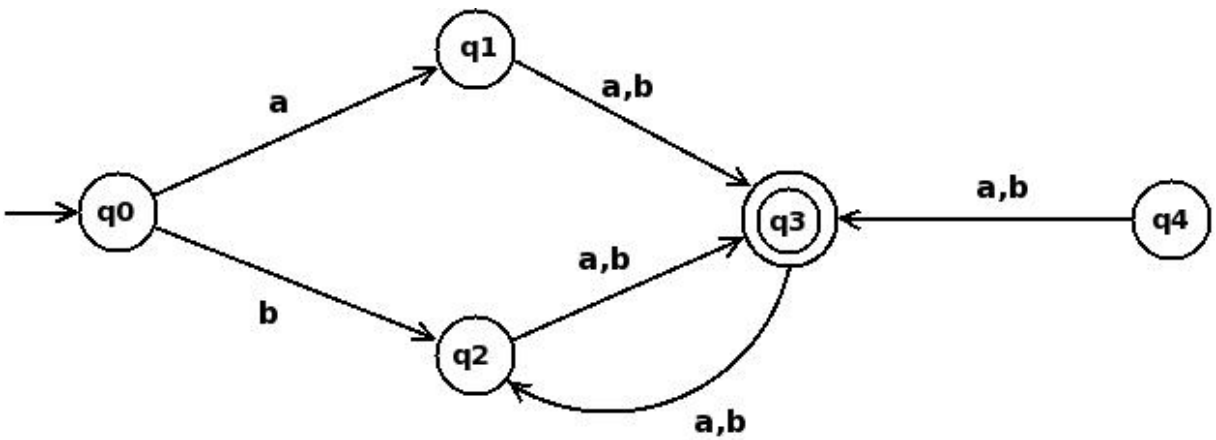


Illustration of minimization algorithm

After eliminating unreachable states:

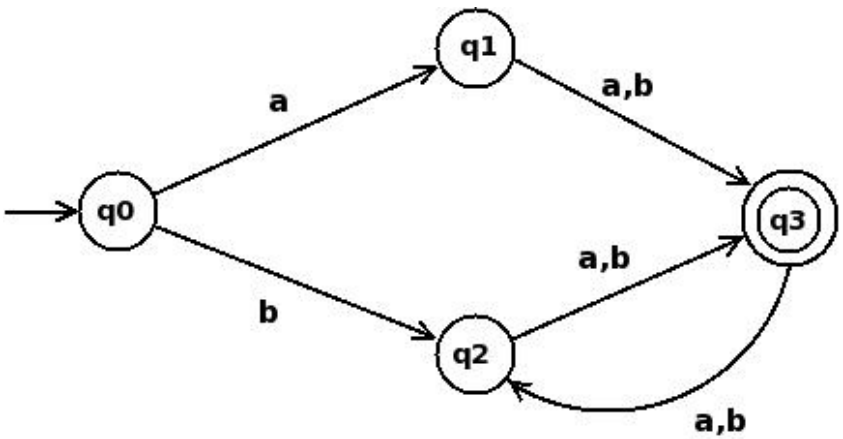
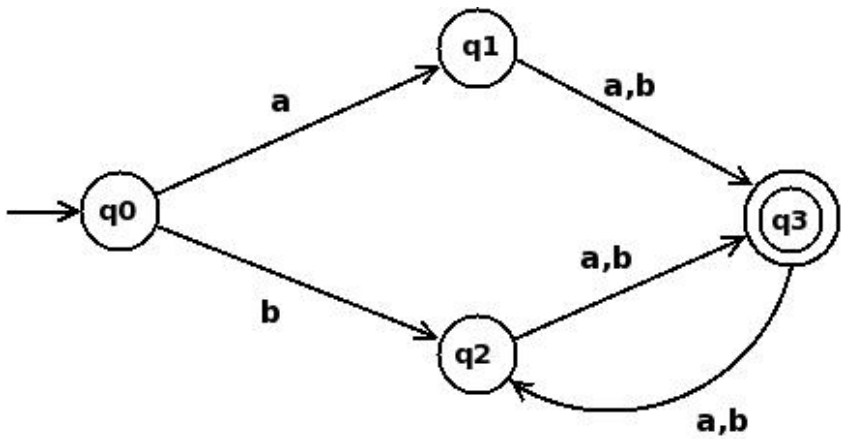


Illustration of minimization algorithm

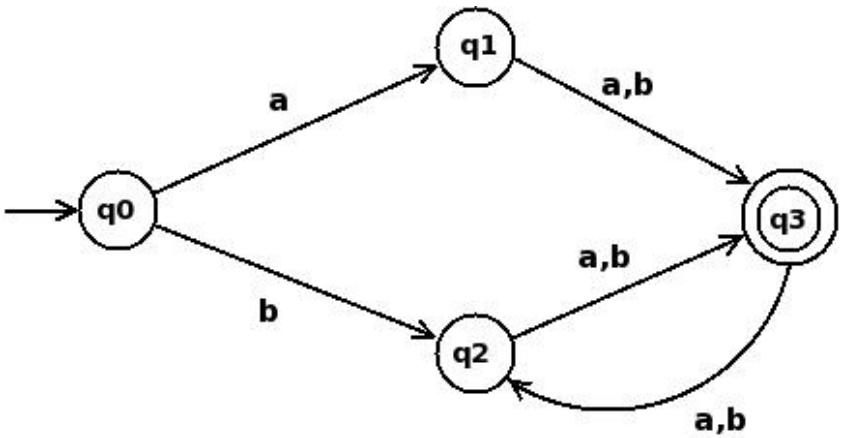
We mark states to be kept distinct using a half matrix:



q0				
q1	.			
q2	.	.		
q3	.	.	.	
	q0	q1	q2	q3

Illustration of minimization algorithm

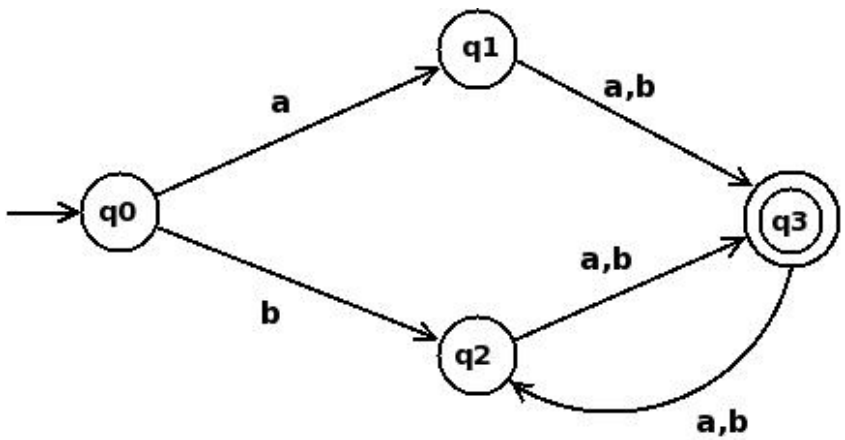
First mark accepting/non-accepting pairs:



q0				
q1	·			
q2	·	·		
q3	✓	✓	✓	
	q0	q1	q2	q3

Illustration of minimization algorithm

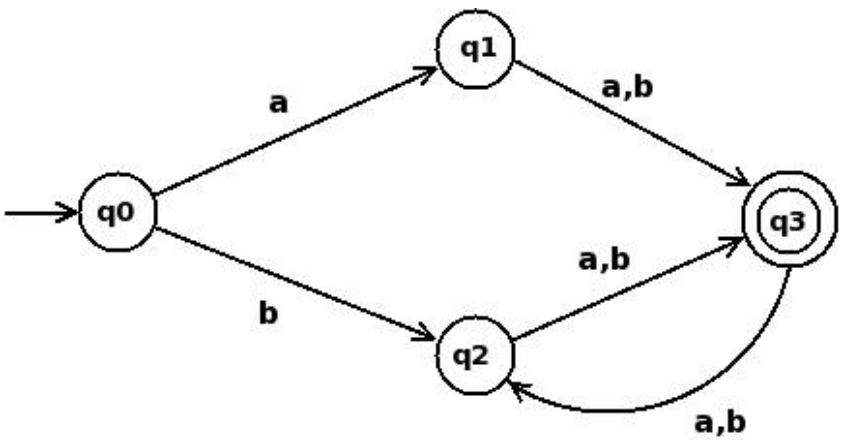
(q0,q1) is unmarked, qo $\xrightarrow{a}$ q1, q1 $\xrightarrow{a}$ q3, and (q1,q3) is marked.



q0				
q1	.			
q2	.	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

Illustration of minimization algorithm

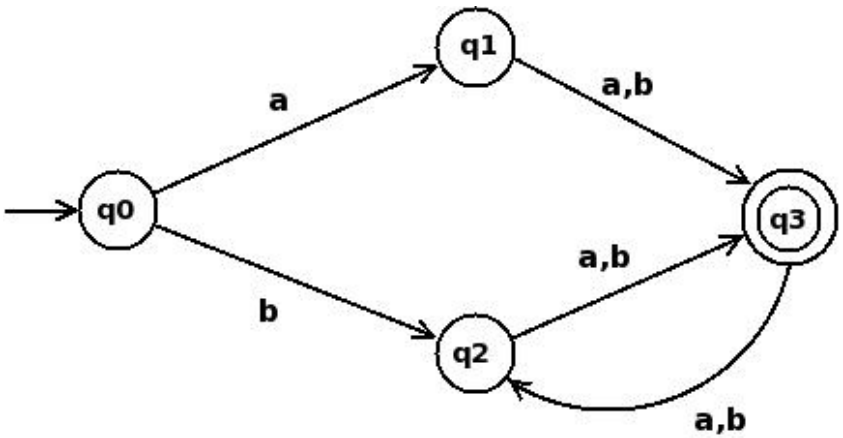
So mark (q0,q1).



q0				
q1	✓			
q2	.	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

Illustration of minimization algorithm

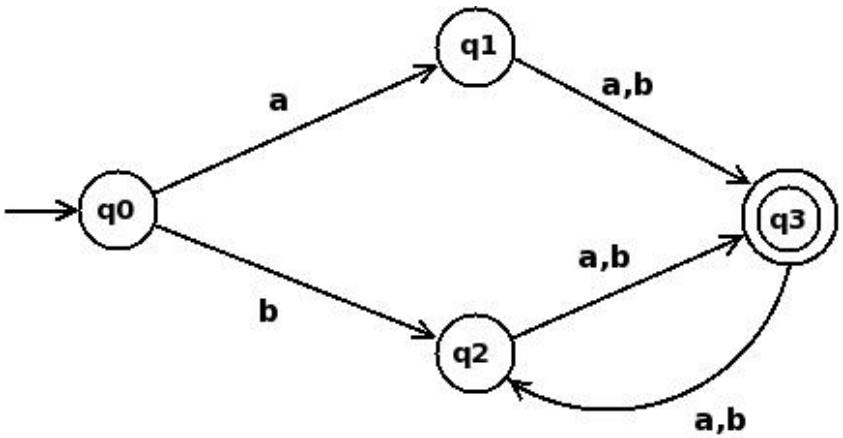
(q0,q2) is unmarked, qo $\xrightarrow{a}$ q1, q2 $\xrightarrow{a}$ q3, and (q1,q3) is marked.



q0				
q1	✓			
q2	.	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

Illustration of minimization algorithm

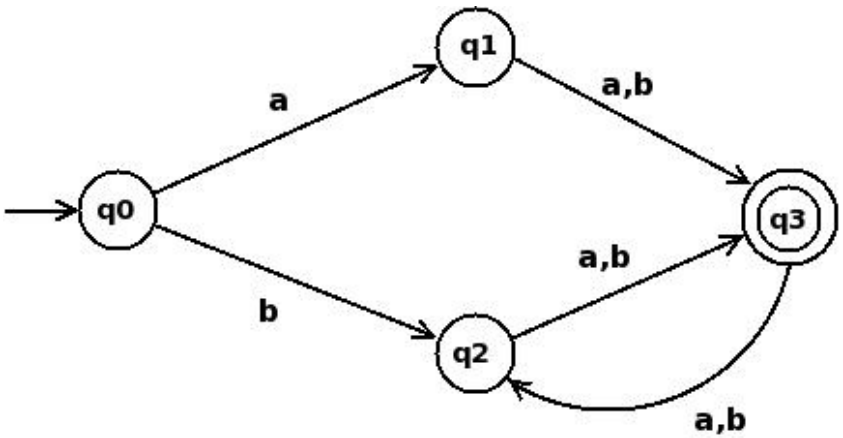
So mark (q0,q2).



q0				
q1	✓			
q2	✓	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

Illustration of minimization algorithm

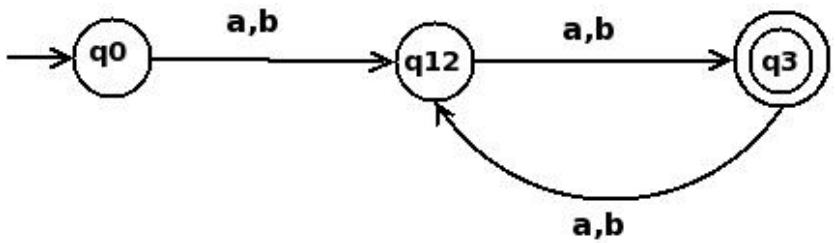
The only remaining unmarked pair (q1,q2) stays unmarked.



q0				
q1	✓			
q2	✓	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

Illustration of minimization algorithm

So obtain minimized DFA by collapsing q1, q2 to a single state.



Reading

Relevant reading:

- Subset Construction: Kozen chapter 6
- Closure properties of regular languages: Kozen chapter 4.
- Minimization: Kozen chapters 13–14.

Next time:

- Regular expressions and Kleene's Theorem.
  (Kozen chapters 7, 8.)