Finite Automata

Informatics 2A: Lecture 3

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1. Languages and Automata
   - What is a 'language'?
   - Finite automata: recap
2. Some formal definitions
   - Finite automaton
   - Regular language
   - DFAs and NFAs
3. Determinization
   - Execution of NFAs
   - The subset construction

Languages and alphabets

Throughout this course, languages will consist of finite sequences of symbols drawn from some given alphabet.

An alphabet Σ is simply some finite set of *letters* or *symbols* which we treat as 'primitive'. These might be ...

- English letters: $\Sigma = \{a, b, \ldots, z\}$
- Decimal digits: $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- ASCII characters: $\Sigma = \{0, 1, \ldots, a, b, \ldots, ?, !, \ldots\}$
- Programming language 'tokens': $\Sigma = \{\texttt{if}, \texttt{while}, \texttt{x}, ==, \ldots\}$
- Words in (some fragment of) a natural language.
- 'Primitive' actions performable by a machine or system, e.g. $\Sigma = \{\texttt{insert50p}, \texttt{pressButton1}, \ldots\}$

In toy examples, we'll use simple alphabets like $\{0, 1\}$ or $\{a, b, c\}$.

What is a 'language'?

A language over an alphabet Σ will consist of finite sequences (strings) of elements of Σ. E.g. the following are strings over the alphabet $\Sigma = \{a, b, c\}$:

$$a \qquad b \qquad ab \qquad cab \qquad bacca \qquad cccccccc$$

There's also the empty string , which we usually write as ϵ. (Note that ϵ isn't itself a symbol in the alphabet!)

A language over Σ is simply a (finite or infinite) set of strings over Σ. A string s is legal in the language L if and only if $s \in L$.

We write Σ^* for the set of *all* possible strings over Σ. So a language L is simply a subset of Σ^*. ($L \subseteq \Sigma^*$)

(N.B. This is just a technical definition — any *real* language is obviously much more than this!)

Ways to define a language

There are many ways in which we might formally define a language:

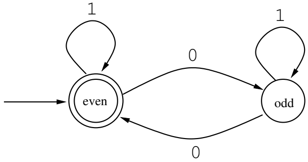
- Direct mathematical definition, e.g.

$$
\begin{aligned}
L_1 &= \{a, aa, ab, abbc\} \\
L_2 &= \{axb \mid x \in \Sigma^*\} \\
L_3 &= \{a^n b^n \mid n \geq 0\}
\end{aligned}
$$

- Regular expressions (see Lecture 5): e.g. $a(a+b)^*b$.
- Formal grammars (see Lecture 9 onwards): e.g. $S \to \epsilon \mid aSb$.
- Specify some machine that tests if a string is legal or not.

The more complex the language, the more complex the machine might need to be. As we shall see, each level in the Chomsky hierarchy is correlated with a certain class of machines.

Finite automata (a.k.a. finite state machines)



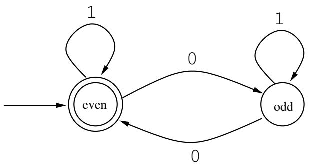
This is an example of a finite automaton over $\Sigma = \{0, 1\}$.

At any moment, the machine is in one of 2 states. From any state, each symbol in Σ determines a 'destination' state we can jump to.

The state marked with the in-arrow is picked out as the starting state. So any string in Σ^* gives rise to a sequence of states.

Certain states (with double circles) are designated as accepting. We call a string 'legal' if it takes us from the start state to some accepting state. In this way, the machine defines a language $L \subseteq \Sigma^*$: the language L is the set of all legal strings.

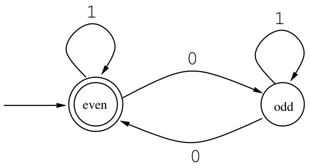
Quick test question . . .



For the finite state machine shown here, which of the following strings are legal (i.e. accepted)?

1. ϵ
2. 11
3. 1010
4. 1101

Quick test question ...

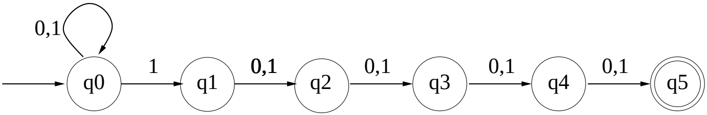


For the finite state machine shown here, which of the following strings are legal (i.e. accepted)?

1. ϵ
2. 11
3. 1010
4. 1101

Answer: 1,2,3 are legal, 4 isn't.

More generally, for any current state and any symbol, there may be zero, one or many new states we can jump to.



Here there are two transitions for '1' from q0, and none from q5.

The language associated with the machine is defined to consist of all strings that are accepted under some possible execution run.

The language associated with the example machine above is

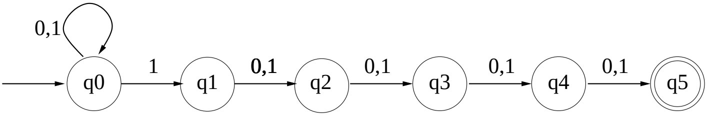
$$\{x \in \Sigma^* \mid \text{the fifth symbol from the end of } x \text{ is } 1\}$$

Formal definition of finite automaton

Formally, a finite automaton with alphabet Σ consists of:

- A finite set Q of states,
- A transition relation $\Delta \subseteq Q \times \Sigma \times Q$,
- A set $S \subseteq Q$ of possible starting states.
- A set $F \subseteq Q$ of accepting states.

Example formal definition



$$Q = \{q0, q1, q2, q3, q4, q5\}$$

$$\begin{aligned}\Delta = \{\,&(q0, 0, q0), (q0, 1, q0), (q0, 1, q1), (q1, 0, q2),\\ &(q1, 1, q2), (q2, 0, q3), (q2, 1, q3), (q3, 0, q4),\\ &(q3, 1, q4), (q4, 0, q5), (q4, 1, q5)\,\}\end{aligned}$$

$$S = \{q0\}$$

$$F = \{q5\}$$

Regular language

Suppose $M = (Q, \Delta, S, F)$ is a finite automaton with alphabet Σ.

We say that a string $x \in \Sigma^*$ is accepted if there exists a path through the set of states Q, starting at some state $s \in S$, ending at some state $f \in F$, with each step taken from the Δ relation, and with the path as a whole spelling out the string x.

This enables us to define the language accepted by M:

$$\mathcal{L}(M) \;=\; \{x \in \Sigma^* \mid x \text{ is accepted by } M\}$$

We call a language $L \subseteq \Sigma^*$ regular if $L = \mathcal{L}(M)$ for *some* finite automaton M.

Regular languages are the subject of lectures 4–8 of the course.

DFAs and NFAs

A finite automaton with alphabet Σ is deterministic if:

- It has exactly one starting state.
- For every state $q \in Q$ and symbol $a \in \Sigma$ there is exactly one state q' for which there exists a transition $q \xrightarrow{a} q'$ in Δ.

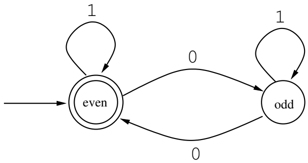
The first condition says that S is a singleton set.
The second condition says that Δ specifies a function $Q \times \Sigma \to Q$.

Deterministic finite automata are usually abbreviated DFAs.

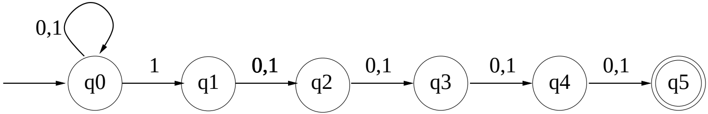
General finite automata are usually called nondeterministic, by way of contrast, and abbreviated NFAs.

Note that every DFA is an NFA.

Example



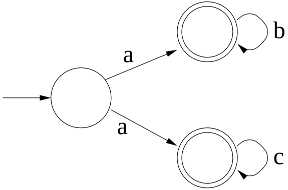
This is a DFA (and hence an NFA).



This is an NFA but not a DFA.

Challenge question

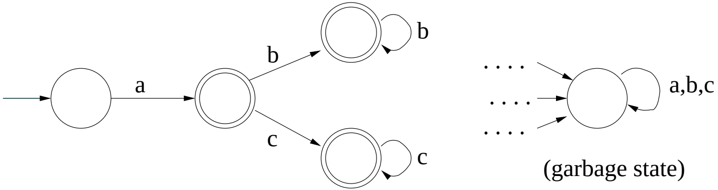
Consider the following NFA over $\{a, b, c\}$:



What is the *minimum* number of states of an equivalent DFA?

Solution

An equivalent DFA must have at least 5 states!



Specifying a DFA

Clearly, a DFA with alphabet Σ can equivalently be given by:

- A finite set Q of states,
- A transition function $\delta : Q \times \Sigma \to Q$,
- A single designated starting state $s \in Q$,
- A set $F \subseteq Q$ of accepting states.

Example:

$$
\begin{array}{rcl}
Q & = & \{\text{even}, \text{odd}\} \\
\delta & : & \begin{array}{r|cc}
 & 0 & 1 \\
\hline
\text{even} & \text{odd} & \text{even} \\
\text{odd} & \text{even} & \text{odd}
\end{array} \\
s & = & \text{even} \\
F & = & \{\text{even}\}
\end{array}
$$

Running a finite automaton

DFAs are dead easy to implement and efficient to run. We don't need much more than a two-dimensional array for the transition function δ. Given an input string x it is easy to follow the unique path determined by x and so determine whether or not the DFA accepts x.

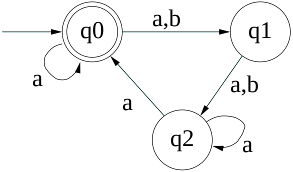
It is by no means so obvious how to run an NFA over an input string x. How do we prevent ourselves from making incorrect nondeterministic choices?

Solution: At each stage in processing the string, keep track of all the states the machine might possibly be in.

Executing an NFA: example

Given an NFA N over Σ and a string $x \in \Sigma^*$, how can we *in practice* decide whether $x \in \mathcal{L}(N)$?

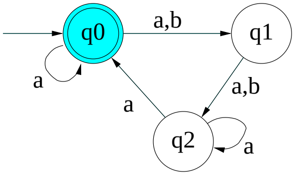
We illustrate with the running example below.



String to process: aba

Stage 0: initial state

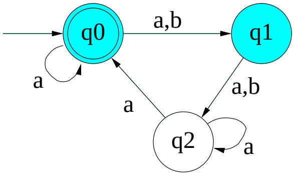
At the start, the NFA *can only be* in the initial state q0.



String to process:	aba
Processed so far:	ϵ
Next symbol:	a

Stage 1: after processing 'a'

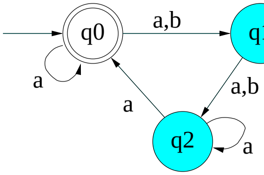
The NFA could now be in either q0 or q1.



String to process:	aba
Processed so far:	a
Next symbol:	b

Stage 2: after processing ‘ab’

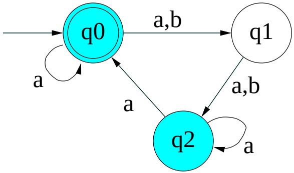
The NFA could now be in either q1 or q2.



String to process:	aba
Processed so far:	ab
Next symbol:	a

Stage 3: final state

The NFA could now be in q2 or q0. (It could have got to q2 in two different ways, though we don't need to keep track of this.)



String to process: aba
Processed so far: aba

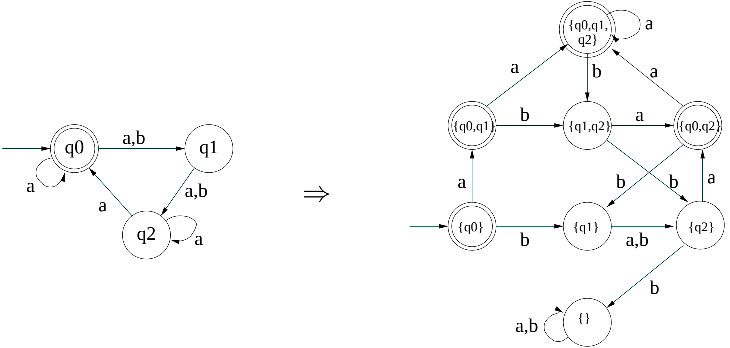
Since we've reached the end of the input string, and the set of possible states includes the accepting state q0, we can say that the string aba is accepted by this NFA.

The key insight

- The process we've just described is a completely deterministic process! Given any current set of 'coloured' states, and any input symbol in Σ, there's only one right answer to the question: 'What should the new set of coloured states be?'
- What's more, it's a finite state process. A 'state' is simply a choice of 'coloured' states in the original NFA N.
  If N has n states, there are 2^n such choices.
- This suggests how an NFA with n states can be converted into an equivalent DFA with 2^n states.

The subset construction: example

Our 3-state NFA gives rise to a DFA with $2^3 = 8$ states. The states of this DFA are subsets of $\{q0, q1, q2\}$.



The accepting states of this DFA are exactly those that *contain* an accepting state of the original NFA.

The subset construction in general

Given an NFA $N = (Q, \Delta, S, F)$, we can define an equivalent DFA $M = (Q', \delta', s', F')$ (over the same alphabet Σ) like this:

- Q' is 2^Q, the set of all subsets of Q. (Also written $\mathcal{P}(Q)$.)
- $\delta'(A, u) = \{q' \in Q \mid \exists q \in A.\ (q, u, q') \in \Delta\}$. (Set of all states reachable via u from *some* state in A.)
- $s' = S$.
- $F' = \{A \subseteq Q \mid \exists q \in A.\ q \in F\}$.

It's then not hard to prove mathematically that $\mathcal{L}(M) = \mathcal{L}(N)$. (See Kozen for details.)

This process is called determinization.

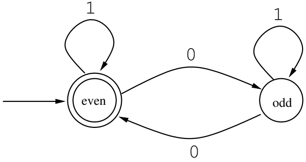
Coming up in lecture 6: Application of this process to efficient string searching.

Summary

- We've shown that for any NFA N, we can construct a DFA M with the same associated language.
- Since every DFA is also an NFA, the classes of languages recognised by DFAs and by NFAs coincide — these are the regular languages.
- Often a language can be specified more concisely by an NFA than by a DFA.
- We can automatically convert an NFA to a DFA, at the risk of an exponential blow-up in the number of states.
- To determine whether a string x is accepted by an NFA, we don't actually need to construct the entire DFA. Instead, we efficiently simulate the execution of the DFA on x on a step-by-step basis. (This is called just-in-time simulation.)

End-of-lecture question 1

Let M be the DFA shown earlier:



Give a simple, concise description of the strings that are in $\mathcal{L}(M)$.

End-of-lecture question 1

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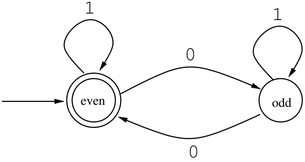

Give a simple, concise description of the strings that are in $\mathcal{L}(M)$.

Answer: They're the strings containing an even number of 0's.

End-of-lecture question 2

Which of these three languages do you think are regular?

$$
\begin{aligned}
L_1 &= \{a, aa, ab, abbc\} \\
L_2 &= \{axb \mid x \in \Sigma^*\} \\
L_3 &= \{a^n b^n \mid n \geq 0\}
\end{aligned}
$$

If not regular, can you explain why not?

End-of-lecture question 2

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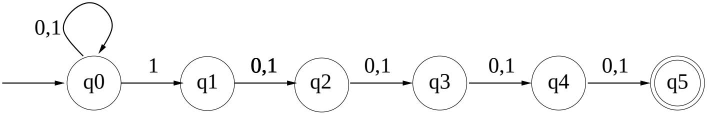
$$\begin{array}{rcl} L_1 & = & \{a, aa, ab, abbc\} \\ L_2 & = & \{axb \mid x \in \Sigma^*\} \\ L_3 & = & \{a^n b^n \mid n \geq 0\} \end{array}$$

If not regular, can you explain why not?

Answer: L_1 is regular (easy to see that any finite set of strings is a regular language). L_2 is regular (easy to give a DFA). L_3 is *not* regular — for the reason, see Lecture 8.

End-of-lecture challenge question 3

Consider our first example NFA over $\{0,1\}$:



What is the number of states of the smallest DFA that recognises the same language?

Answer given at end of Lecture 4.

Reference material

- Kozen chapters 3, 5 and 6.
- J & M section 2.2 (rather brief).