Regular expressions and Kleene's theorem

Informatics 2A: Lecture 5

Alex Simpson

School of Informatics
University of Edinburgh
als@inf.ed.ac.uk

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    - Kleene's theorem
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A simple application of NFAs

Consider the following little theorem:

If L_1 and L_2 are regular languages over Σ, so is $L_1 \cup L_2$.

This *can* be shown using DFAs ... but it's dead easy using NFAs.

Suppose $N_1 = (Q_1, \Delta_1, S_1, F_1)$ is an NFA for L_1, and $N_2 = (Q_2, \Delta_2, S_2, F_2)$ is an NFA for L_2.

We may assume $Q_1 \cap Q_2 = \emptyset$ (just relabel states if not).

Now consider the NFA

$$(Q_1 \cup Q_2,\ \Delta_1 \cup \Delta_2,\ S_1 \cup S_2,\ F_1 \cup F_2)$$

This is just N_1 and N_2 'side by side'. Clearly, this NFA recognizes precisely $L_1 \cup L_2$.

(Quite useful in practice — no state explosion!)

Closure properties of regular languages

- We've seen that if both L_1 and L_2 are regular languages, so is $L_1 \cup L_2$.
- We sometimes express this by saying that regular languages are closed under the 'union' operation. ('Closed' used here in the sense of 'self-contained'.)
- We will show that regular languages are closed under other operations too: Concatenation: $L_1.L_2$ and Kleene star: L^*

  For these, it will be convenient to work with a minor variation on NFAs: ϵ-NFAs.
- All this will lead us to another (and very useful!) way of defining regular languages: via regular expressions.

Concatenation and Kleene star

- Concatenation: write $L_1.L_2$ for the language

$$\{xy \mid x \in L_1, y \in L_2\}$$

E.g. if $L_1 = \{aaa\}$ and $L_2 = \{b, c\}$ then $L_1.L_2$ is the language $\{aaab,\ aaac\}$.

- Kleene star: let L^* denote the language

$$\{\epsilon\} \cup L \cup L.L \cup L.L.L \cup \ldots$$

E.g. if $L_3 = \{aaa, b\}$ then L_3^* contains strings like *aaaaaa*, *bbbbb*, *baaaaaabbaaa*, etc.
More precisely, L_3^* contains all strings over $\{a, b\}$ in which the letter a always appears in sequences of length some multiple of 3

Clicker question

Consider the language over the alphabet $\{a, b, c\}$

$$L = \{x \mid x \text{ starts with } a \text{ and ends with } c\}$$

Which of the following strings is *not* valid for the language $L.L$?

1. $abcabc$
2. $acacac$
3. $abcbcac$
4. $abcbacbc$

Clicker question

Consider the (same) language over the alphabet $\{a, b, c\}$

$$L = \{x \mid x \text{ starts with } a \text{ and ends with } c\}$$

Which of the following strings is *not* valid for the language L^* ?

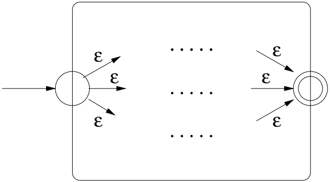
1. ϵ
2. $acaca$
3. $abcbc$
4. $acacacacac$

NFAs with ϵ-transitions

We can vary the definition of NFA by also allowing transitions labelled with the special symbol ϵ (*not* a symbol in Σ).

The automaton may (but doesn't have to) perform an ϵ-transition at any time, without reading an input symbol.

This is quite convenient: for instance, we can turn any NFA into an ϵ-NFA with just one start state and one accepting state:



(Add ϵ-transitions from new start state to each state in S, and from each state in F to new accepting state.)

Equivalence to ordinary NFAs

Allowing ϵ-transitions is just a convenience: it doesn't fundamentally change the power of NFAs.

If $N = (Q, \Delta, S, F)$ is an ϵ-NFA, we can convert N to an ordinary NFA with the same associated language, by simply 'expanding' Δ and S to allow for silent ϵ-transitions.

Formally, the ϵ-closure of a transition relation $\Delta \subseteq Q \times (\Sigma \cup \{\epsilon\}) \times Q$ is the smallest relation $\overline{\Delta}$ that contains Δ and satisfies:

- if $(q, u, q') \in \overline{\Delta}$ and $(q', \epsilon, q'') \in \Delta$ then $(q, u, q'') \in \overline{\Delta}$;
- if $(q, \epsilon, q') \in \Delta$ and $(q', u, q'') \in \overline{\Delta}$ then $(q, u, q'') \in \overline{\Delta}$.

Likewise, the ϵ-closure of S under Δ is the smallest set of states $\overline{S}_\Delta$ that contains S and satisfies:

- if $q \in \overline{S}_\Delta$ and $(q, \epsilon, q') \in \Delta$ then $q' \in \overline{S}_\Delta$.

We can then replace the ϵ-NFA (Q, Δ, S, F) with the ordinary NFA

$$(Q,\ \overline{\Delta} \cap (Q \times \Sigma \times Q),\ \overline{S}_\Delta,\ F)$$

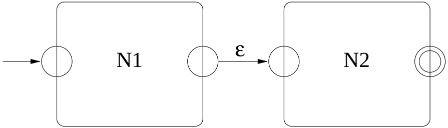
Concatenation of regular languages

We can use ϵ-NFAs to show that regular languages are closed under the concatenation operation:

$$L_1.L_2 \;=\; \{xy \mid x \in L_1, y \in L_2\}$$

If L_1, L_2 are any regular languages, choose ϵ-NFAs N_1, N_2 that define them. As noted earlier, we can pick N_1 and N_2 to have just one start state and one accepting state.

Now hook up N_1 and N_2 like this:



Clearly, this NFA corresponds to the language $L_1.L_2$.

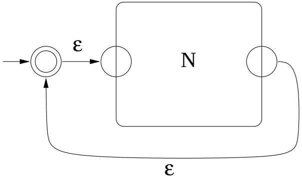
To ponder: do we need the ϵ-transition in the middle?

Kleene star

Similarly, we can now show that regular languages are closed under the Kleene star operation:

$$L^* \;=\; \{\epsilon\} \cup L \cup L.L \cup L.L.L \cup \ldots$$

For suppose L is represented by an ϵ-NFA N with one start state and one accepting state. Consider the following ϵ-NFA:



Clearly, this ϵ-NFA corresponds to the language L^*.

Regular expressions

We've been looking at ways of specifying regular languages via machines (often presented as pictures). But it's also useful to have more textual ways of defining languages.

A regular expression is a written mathematical expression that defines a language over a given alphabet Σ.

- The basic regular expressions are

$$\emptyset \qquad \epsilon \qquad a \quad (\text{for } a \in \Sigma)$$

- From these, more complicated regular expressions can be built up by (repeatedly) applying the two binary operations $+$, $.$ and the unary operation * . Example: $(a.b+\epsilon)^*+a$

We use brackets to indicate precedence. In the absence of brackets, * binds more tightly than ., which itself binds more tightly than $+$.

$$\text{So} \quad a+b.a^* \quad \text{means} \quad a+(b.(a^*))$$

Also the dot is often omitted: ab means $a.b$

How do regular expressions define languages?

A regular expression is itself just a written expression (actually in some context-free 'meta-language'). However, every regular expression α over Σ can be seen as defining an actual language $\mathcal{L}(\alpha) \subseteq \Sigma^*$ in the following way:

- $\mathcal{L}(\boldsymbol{\emptyset}) = \emptyset, \quad \mathcal{L}(\boldsymbol{\epsilon}) = \{\epsilon\}, \quad \mathcal{L}(a) = \{a\}.$
- $\mathcal{L}(\alpha + \beta) = \mathcal{L}(\alpha) \cup \mathcal{L}(\beta)$
- $\mathcal{L}(\alpha.\beta) = \mathcal{L}(\alpha) \,.\, \mathcal{L}(\beta)$
- $\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^*$

Example: $a + ba^*$ defines the language $\{a, b, ba, baa, baaa, \ldots\}$.

The languages defined by $\boldsymbol{\emptyset}, \boldsymbol{\epsilon}, a$ are obviously regular.

What's more, we've seen that regular languages are closed under union, concatenation and Kleene star.

This means every regular expression defines a regular language. (Proof by induction on the size of the regular expression.)

Clicker question

Consider (again) the language

$$\{x \in \{0,1\}^* \mid x \text{ contains an even number of 0's}\}$$

Which of the following regular expressions is *not* a possible definition of this language?

1. $(1^*01^*01^*)^*$
2. $(1^*01^*0)^*1^*$
3. $1^*(01^*0)^*1^*$
4. $(1 + 01^*0)^*$

Kleene's theorem

We've seen that every regular expression defines a regular language.

The main goal of today's lecture is to show the converse, that every regular language can be defined by a regular expression. For this purpose, we introduce Kleene algebra: the algebra of regular expressions.

The equivalence between regular languages and expressions is: Kleene's theorem

> *DFAs and regular expressions give rise to exactly the same class of languages (the regular languages).*

As we've already seen, NFAs (with or without ϵ-transitions) also give rise to this class of languages.

So the evidence is mounting that the class of regular languages is mathematically a very 'natural' class to consider.

Kleene algebra

Regular expressions give a textual way of specifying regular languages. This is useful e.g. for communicating regular languages to a computer.

Another benefit: regular expressions can be manipulated using algebraic laws (Kleene algebra). For example:

$$\begin{array}{rclcrcl}
\alpha + (\beta + \gamma) & = & (\alpha + \beta) + \gamma & \quad & \alpha + \beta & = & \beta + \alpha \\
\alpha + \emptyset & = & \alpha & & \alpha + \alpha & = & \alpha \\
\alpha(\beta\gamma) & = & (\alpha\beta)\gamma & & \epsilon\alpha & = & \alpha\epsilon \ = \ \alpha \\
\alpha(\beta + \gamma) & = & \alpha\beta + \alpha\gamma & & (\alpha + \beta)\gamma & = & \alpha\gamma + \beta\gamma \\
\emptyset\alpha & = & \alpha\emptyset \ = \ \emptyset & & \epsilon + \alpha\alpha^* & = & \epsilon + \alpha^*\alpha = \alpha^*
\end{array}$$

Often these can be used to simplify regular expressions down to more pleasant ones.

Other reasoning principles

Let's write $\alpha \leq \beta$ to mean $\mathcal{L}(\alpha) \subseteq \mathcal{L}(\beta)$ (or equivalently $\alpha + \beta = \beta$). Then

$$\begin{aligned} \alpha\gamma + \beta \leq \gamma \quad &\Rightarrow \quad \alpha^*\beta \leq \gamma \\ \beta + \gamma\alpha \leq \gamma \quad &\Rightarrow \quad \beta\alpha^* \leq \gamma \end{aligned}$$

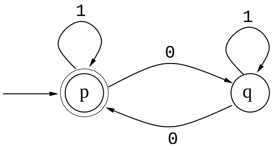
Arden's rule: Given an equation of the form $X = \alpha X + \beta$, its smallest solution is $X = \alpha^*\beta$.

What's more, if $\epsilon \notin \mathcal{L}(\alpha)$, this is the *only* solution.

Intriguing fact: The rules on this slide and the last form a complete set of reasoning principles, in the sense that if $\mathcal{L}(\alpha) = \mathcal{L}(\beta)$, then '$\alpha = \beta$' is provable using these rules. (Beyond scope of Inf2A.)

DFAs to regular expressions

We use an example to show how to convert a DFA to an equivalent regular expression.



For each state r, let the variable X_r stand for the set of strings that take us from r to an accepting state. Then we can write some simultaneous equations:

$$
\begin{aligned}
X_p &= 1X_p + 0X_q + \epsilon \\
X_q &= 1X_q + 0X_p
\end{aligned}
$$

We solve the equations by eliminating one variable at a time:

$$
\begin{array}{rrcl}
 & X_q & = & 1^*0X_p \quad \text{by Arden's rule} \\
\text{So} & X_p & = & 1X_p + 01^*0X_p + \epsilon \\
 & & = & (1 + 01^*0)X_p + \epsilon \\
\text{So} & X_p & = & (1 + 01^*0)^* \quad \text{by Arden's rule}
\end{array}
$$

Since the start state is p, the resulting regular expression for X_p is the one we are seeking. Thus the language recognised by the automaton is:

$$(1 + 01^*0)^*$$

The method we have illustrated here, in fact, works for arbitrary NFAs (without ϵ-transitions).

Theory of regular languages: overview

$$\begin{tikzcd}
\text{DFAs} \arrow[d, "\text{solve equations}"'] & & \text{NFAs} \arrow[ll, "\text{subset construction}"'] \\
\text{reg exps} \arrow[rr, "\text{closure properties}"'] & & \epsilon\text{-NFAs} \arrow[u, "\text{remove } \epsilon\text{-transitions}"']
\end{tikzcd}$$

Reading

Relevant reading:

- Regular expressions: Kozen chapters 7,8; J & M chapter 2.1. (Both texts actually discuss more general 'patterns' — see next lecture.)
- From regular expressions to NFAs: Kozen chapter 8; J & M chapter 2.3.
- Kleene algebra: Kozen chapter 9.
- From NFAs to regular expressions: Kozen chapter 9.

Next time: Some applications of all this theory.

- Pattern matching
- Lexical analysis

Appendix: (non-examinable) proof of Kleene's theorem

Given an NFA $N = (Q, \Delta, S, F)$ (without ϵ-transitions), we'll show how to define a regular expression defining the same language as N.

In fact, to build this up, we'll construct a three-dimensional array of regular expressions α_{uv}^{X}: one for every $u \in Q, v \in Q, X \subseteq Q$.

Informally, α_{uv}^{X} will define the set of *strings that get us from u to v allowing only intermediate states in X*.

We shall build suitable regular expressions $\alpha_{u,v}^{X}$ by working our way from smaller to larger sets X.

Eventually, the language defined by N will be given by the sum $(+)$ of the languages α_{sf}^{Q} for all states $s \in S$ and $f \in F$.

Construction of α_{uv}^X

Here's how the regular expressions α_{uv}^X are built up.

- If $X = \emptyset$, let $a_1, \ldots, a_k$ be all the symbols a such that $(u, a, v) \in \Delta$. Two subcases:
  - If $u \neq v$, take $\alpha_{uv}^{\emptyset} \;=\; a_1 + \cdots + a_k$
  - If $u = v$, take $\alpha_{uv}^{\emptyset} \;=\; (a_1 + \cdots + a_k) + \epsilon$

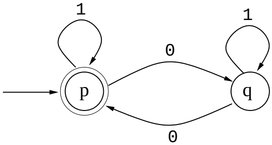
  Convention: if $k = 0$, take '$a_1 + \ldots + a_k$' to mean $\emptyset$.
- If $X \neq \emptyset$, choose any $q \in X$, let $Y = X - \{q\}$, and define

$$\alpha_{uv}^X \;=\; \alpha_{uv}^Y + \alpha_{uq}^Y (\alpha_{qq}^Y)^* \alpha_{qv}^Y$$

Applying these rules repeatedly gives us $\alpha_{u,v}^X$ for every u, v, X.

NFAs to regular expressions: tiny example

Let's revisit our old friend:



Here p is the only start state and the only accepting state.
By the rules on the previous slide:

$$\alpha_{p,p}^{\{p,q\}} = \alpha_{p,p}^{\{p\}} + \alpha_{p,q}^{\{p\}}(\alpha_{q,q}^{\{p\}})^*\alpha_{q,p}^{\{p\}}$$

Now by inspection (or by the rules again), we have

$$\begin{aligned} \alpha_{p,p}^{\{p\}} &= 1^* & \alpha_{p,q}^{\{p\}} &= 1^*0 \\ \alpha_{q,q}^{\{p\}} &= 1+01^*0 & \alpha_{q,p}^{\{p\}} &= 01^* \end{aligned}$$

So the required regular expression is

$$1^* \ + \ 1^*0(1+01^*0)^*01^* \qquad \text{(A bit messy!)}$$