Undecidability

Informatics 2A: Lecture 30

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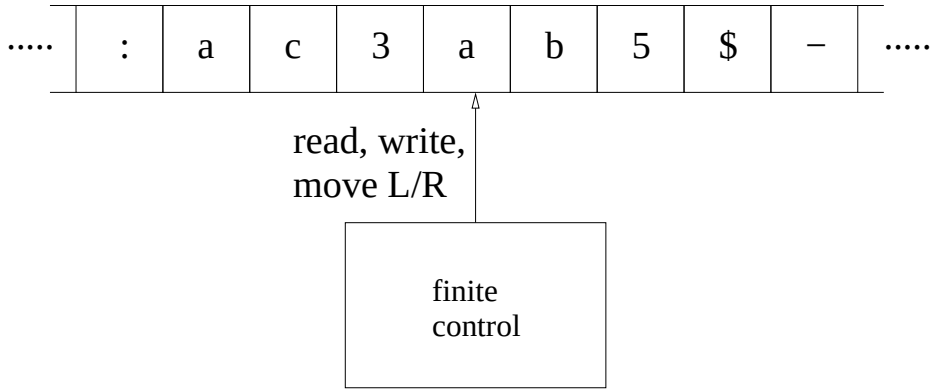
Recap: Turing machines



- If $|\Sigma| \geq 2$, any kind of 'finite data' can be coded up as a string in Σ^*, which can then be written onto a Turing machine tape. (E.g. natural numbers could be written in binary.)
- According to the Church-Turing thesis (CTT), any 'mechanical computation' that can be performed on finite data can be performed in principle by a Turing machine.
- Any decent programming language has the same computational power in principle as a Turing machine.

Universal Turing machines

Consider any Turing machine with input alphabet Σ.

Such a machine T is itself specified by a finite amount of information, so can in principle be 'coded up' by a string $\overline{T} \in \Sigma^*$. (Details don't matter).

So one can imagine a universal Turing machine U which:

- Takes as its input a coded description $\overline{T}$ of some TM T, along with an input string s, separated by a blank symbol.
- Simulates the behaviour of T on the input string s. (N.B. a single step of T may require many steps of U.)
  - If T ever halts (i.e. enters final state), U will halt.
  - If T runs forever, U will run forever.

If we believe CTT, such a U must exist — but in any case, it's possible to construct one explicitly.

The concept of a general-purpose computer

Alan Turing's discovery of the existence of a universal Turing machine (1936) was in some sense the fundamental insight that gave us the general-purpose (programmable) computer!

In most areas of life, we have different machines for different jobs. So isn't it remarkable that a single physical machine can be persuaded to perform as many different tasks as a computer can . . . just by feeding it with a cunning sequence of 0's and 1's!

The halting problem

The universal machine U in effect serves as a recognizer for the set

$$\{\overline{T}\_s \mid T \text{ halts on input } s\}$$

But is there also a machine V that recognizes the set

$$\{\overline{T}\_s \mid T \text{ doesn't halt on input } s\} \ ?$$

If there were, then given any T and s, we could run U and V in parallel, and we'd eventually get an answer to the question "does T halt on input s?"

Conversely, if there were a machine that answered this question, we could construct a machine V with the above property.

Theorem: There is no such Turing machine V!
So the halting problem is undecidable.

Proof of undecidability

Why is the halting problem undecidable?

Suppose V existed. Then we could easily make a Turing machine W that recognised the set L defined by:

$$L = \{s \in \Sigma^* \mid \text{ the TM coded by } s \text{ runs forever on the input } s\}$$

(W could just write two copies of its input string s, separated by a blank, and thereafter behave as V.)

Now consider what W does when given the string $\overline{W}$ as input. That is, the input to W is the string that encodes W itself.

- W accepts $\overline{W}$ iff W runs forever on $\overline{W}$ (since W recognises L)
- but W accepts $\overline{W}$ iff W halts on $\overline{W}$ (definition of acceptance)

Contradiction!!! So V can't exist after all!

Precursor: Russell's paradox (1901)

Define R to be the set of all sets that don't contain themselves:

$$R = \{S \mid S \notin S\}$$

Does R contain itself, i.e. is $R \in R$?

Russell's analogy: The village barber shaves exactly those men in the village who don't shave themselves. Does the barber shave himself, or not?

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Russell's analogy: The village barber shaves exactly those men in the village who don't shave themselves. Does the barber shave himself, or not?

Conclusion: no man exists in the village with the property identified by Russell.

Decidable vs. semidecidable sets

In general, a set S (e.g. $\subseteq \Sigma^*$) is called decidable if there's a mechanical procedure which, given $s \in \Sigma^*$, will always return a yes/no answer to the question "Is $s \in S$?".
E.g. the set $\{s \mid s \text{ represents a prime number}\}$ is decidable.

We say S is semidecidable if there's a mechanical procedure which will return 'yes' precisely when $s \in S$ (it isn't obliged to return anything if $s \notin S$).

Semidecidable sets coincide with recursively enumerable sets, i.e. those that can be 'listed' by a mechanical procedure left to run forever. Also with recursively enumerable (i.e., Type 0) languages as defined in lectures 28–9

The halting set $\{\overline{T}\_ s \mid T \text{ halts on input } s\}$ is an example a semidecidable set that isn't decidable. So there exist Type 0 languages for which membership is undecidable.

Separating Type 0 and Type 1

Every Type 1 (context-sensitive) language is decidable.
(The argument was outlined in Lecture 29, Slide 12.)

As we have seen, the halting set

$$\{\overline{T}\_s \mid T \text{ halts on input } s\}$$

is an undecidable Type 0 language.

So the halting set is an example of a Type 0 language that is not a Type 1 language.

Undecidable problems in mathematics

The existence of 'mechanically unsolvable' mathematical problems was in itself a major breakthrough in mathematical logic: until about 1930, some people (the influential mathematician David Hilbert, in particular) hoped there might be a single killer algorithm that could solve 'all' mathematical problems!

Once we have one example of an unsolvable problem (the halting problem), we can use it to obtain others — typically by showing "the halting problem can be reduced to problem X."
(If we had a mechanical procedure for solving X, we could use it to solve the halting problem.)

Example: Provability of theorems

Let M be some reasonable (consistent) formal logical system for proving mathematical theorems (something like Peano arithmetic or Zermelo-Fraenkel set theory).

Theorem: The set of theorems provable in M is semidecidable (and hence is a Type 0 language), but not decidable.

Proof: Any reasonable system M will be able to prove all true statements of the form "T halts on input s". So if we could decide M-provability, we could solve the halting problem.

Corollary (Gödel): However strong M is, there are mathematical statements P such that neither P nor $\neg P$ is provable in M.

Proof: Otherwise, given any P we could search through all possible M-proofs until either a proof of P or of $\neg P$ showed up. This would give us an algorithm for deciding M-provability.

Example: Diophantine equations

Suppose we're given a set of simultaneous equations involving polynomials in several variables with integer coefficients. E.g.

$$\begin{aligned} 3xy + 4z + 5wx^2 &= 27 \\ x^2 + y^3 - 9z &= 4 \\ w^5 - z^4 &= 31 \\ x^2 + y^2 + z^2 + w^2 &= 2536427 \end{aligned}$$

Hilbert's 10th Problem (1900): Is there a mechanical procedure for determining whether a set of polynomial equations has an integer solution?

Matiyasevich' Theorem (1970): it is undecidable, whether a set of polynomial equations has an integer solution.

(By contrast, it's decidable whether there's a solution in real numbers!)

Examples from Language Processing itself

(The snake bites its own tail . . .)

- Pretty much all natural problems involving regular languages / DFAs / NFAs are decidable. E.g. "do two DFAs define the same language?": apply the minimization algorithm and see if they're isomorphic.
- This isn't true for context-free languages. E.g. it is undecidable, given a context-free grammar G with terminals Σ, whether or not $\mathcal{L}(G)$ is the whole of Σ^*.
- It is also undecidable, given CFGs G_1 and G_2, whether $\mathcal{L}(G_1) \cap \mathcal{L}(G_2)$ is a context-free language.

So undecidability does crop up 'naturally' in many areas of mathematics.

End-of-course questions

What is the status of determining whether $\mathcal{L}(G)$ is nonempty . . .

Q1: . . . when G is a regular grammar? (not too hard)

1. Decidable
2. Semidecidable
3. Not even semidecidable

End-of-course questions

What is the status of determining whether $\mathcal{L}(G)$ is nonempty ...

Q1: ... when G is a regular grammar? (not too hard)

Q2: ... when G is a context-free grammar? (very hard)

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End-of-course questions

What is the status of determining whether $\mathcal{L}(G)$ is nonempty ...

Q1: ... when G is a regular grammar? (not too hard)

Q2: ... when G is a context-free grammar? (very hard)

Q3: ... when G is a context-sensitive grammar? (fiendishly hard)

1. Decidable
2. Semidecidable
3. Not even semidecidable

End-of-course answers: Questions 1 and 2

Q1: decidable.

Convert the regular grammar G to an NFA M. The language $\mathcal{L}(G)$ is nonempty if and only if there is a path in M from an initial state to a final state. It is easy to see that the existence of such a path is decidable, using a simple search algorithm.

Q2: decidable.

This is proved using the 'tree surgery' used in the proof of the context-free pumping lemma. If there is some syntax tree in the CFG G, then, using tree surgery, there must exist a syntax tree in which no path has a repeated noterminal. Given this, one just needs to search the finite space of all potential such syntax trees to see if one can be constructed in which all leaves are decorated by terminals (or ϵ).

End-of-course answers: Question 3

Q3: semidecidable.

The following algorithm terminates if and only if $\mathcal{L}(G)$ is nonempty. Iterate, in turn, through the infinite set $x_1, x_2, x_3, x_4, \ldots$ of all strings over Σ. For each x_i, perform the algorithm (slides 3 and 12 of Lecture 29) that decides if $x_i \in \mathcal{L}(G)$. As soon as one of the tests $x_i \in \mathcal{L}(G)$ succeeds, stop the process.

It is not however decidable if $\mathcal{L}(G)$ is nonempty, for context sensitive G. Suppose, for contradiction, that this were decidable. Let G' be a context-free grammar. The complement $\Sigma^* - \mathcal{L}(G')$ is context sensitive and we can algorithmically produce a context sensitive grammar G for it. Apply our hypothesised decision procedure to test if $\mathcal{L}(G)$ is nonempty. This returns 'yes' if and only if $\mathcal{L}(G') \neq \Sigma^*$, and 'no' otherwise. By swapping 'yes' and 'no', we have constructed a decision procedure to decide if $\mathcal{L}(G')$ is the whole of Σ^* for context-free G'. This contradicts slide 13.

That's all folks!

That concludes the course syllabus.

On Thursday, John and I will present a joint revision lecture, in which we shall discuss:

- the exam structure
- examinable material
- pointers to UG3 (and upwards) Informatics courses that continue from this one