

# Constructions on Finite Automata

## Informatics 2A: Lecture 4

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23rd September, 2014

- 1 Closure properties of regular languages
  - Union
  - Intersection
  - Complement
  
- 2 DFA minimization
  - The problem
  - An algorithm for minimization

## Recap of Lecture 3

- A language is a set of strings over an alphabet  $\Sigma$ .
- A language is called **regular** if it is recognised by some NFA.
- DFAs are an important subclass of NFAs.
- An NFA with  $n$  states can be **determinized** to an equivalent DFA with  $2^n$  states, using the **subset construction**.
- Therefore the regular languages are exactly the languages recognised by DFAs.

## Union of regular languages

Consider the following little theorem:

*If  $L_1$  and  $L_2$  are regular languages over  $\Sigma$ , so is  $L_1 \cup L_2$ .*

This is **dead easy** to prove using NFAs.

Suppose  $N_1 = (Q_1, \Delta_1, S_1, F_1)$  is an NFA for  $L_1$ , and  $N_2 = (Q_2, \Delta_2, S_2, F_2)$  is an NFA for  $L_2$ .

We may assume  $Q_1 \cap Q_2 = \emptyset$  (just relabel states if not).

Now consider the NFA

$$(Q_1 \cup Q_2, \Delta_1 \cup \Delta_2, S_1 \cup S_2, F_1 \cup F_2)$$

This is just  $N_1$  and  $N_2$  'side by side'. Clearly, this NFA recognizes precisely  $L_1 \cup L_2$ .

Number of states =  $|Q_1| + |Q_2|$  — no state explosion!

# Intersection of regular languages

*If  $L_1$  and  $L_2$  are regular languages over  $\Sigma$ , so is  $L_1 \cap L_2$ .*

Suppose  $N_1 = (Q_1, \Delta_1, S_1, F_1)$  is an NFA for  $L_1$ , and  $N_2 = (Q_2, \Delta_2, S_2, F_2)$  is an NFA for  $L_2$ .

We define a **product** NFA  $(Q', \Delta', S', F')$  by:

$$\begin{aligned} Q' &= Q_1 \times Q_2 \\ (q, r) \xrightarrow{a} (q', r') \in \Delta' &\iff q \xrightarrow{a} q' \in \Delta_1 \text{ and } r \xrightarrow{a} r' \in \Delta_2 \\ S' &= S_1 \times S_2 \\ F' &= F_1 \times F_2 \end{aligned}$$

Number of states =  $|Q_1| \times |Q_2|$  — a bit more costly than union!

If  $N_1$  and  $N_2$  are DFAs then the product automaton is a DFA too.

## Complement of a regular language

( Recall the set-difference operation,

$$A - B = \{x \in A \mid x \notin B\}$$

where  $A, B$  are sets. )

*If  $L$  is a regular language over  $\Sigma$ , then so is  $\Sigma^* - L$ .*

Suppose  $N = (Q, \delta, s, F)$  is a DFA for  $L$ .

Then  $(Q, \delta, s, Q - F)$  is a DFA for  $\Sigma^* - L$ . (We simply swap the accepting and rejecting states in  $N$ .)

Number of states =  $|Q|$  — no blow up at all, **but we are required to start with a DFA**. This in itself has size implications.

The complement construction **does not work** if  $N$  is not deterministic!

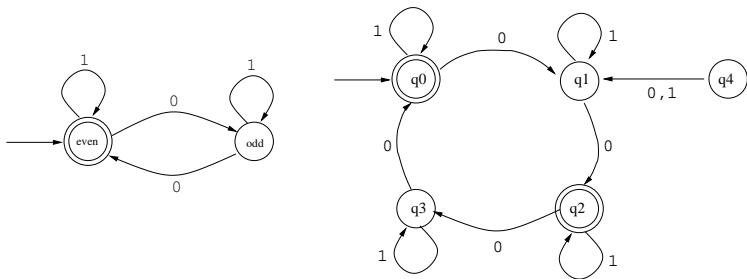
# Closure properties of regular languages

- We've seen that if both  $L_1$  and  $L_2$  are regular languages, then so are:
  - $L_1 \cup L_2$  (union)
  - $L_1 \cap L_2$  (intersection)
  - $\Sigma^* - L_1$  (complement)
- We sometimes express this by saying that regular languages are **closed under** the operations of union, intersection and complementation. ('Closed' used here in the sense of 'self-contained'.)
- Each closure property corresponds to an explicit construction on finite automata. Sometimes this uses NFAs (union), sometimes DFAs (complement), and sometimes the construction works equally well for both NFAs and DFAs (intersection).

# The Minimization Problem

Determinization involves an exponential blow-up in the automaton. Is it sometimes possible to reduce the size of the resulting DFA?

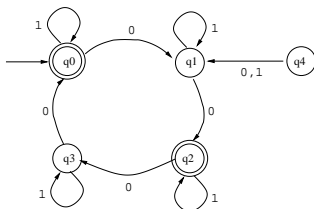
Many different DFAs can give rise to the same language, e.g.:



We shall see that there is always a **unique smallest** DFA for a given regular language.



## DFA minimization



We perform the following steps to 'reduce'  $M$  above:

- Throw away **unreachable** states (in this case,  $q_4$ ).
- Squish together **equivalent** states, i.e. states  $q, q'$  such that: every string accepted starting from  $q$  is accepted starting from  $q'$ , and vice versa. (In this case,  $q_0$  and  $q_2$  are equivalent, as are  $q_1$  and  $q_3$ .)

Let's write  $\text{Min}(M)$  for the resulting reduced DFA. In this case,  $\text{Min}(M)$  is essentially the two-state machine on the previous slide.

## Properties of minimization

The minimization operation on DFAs enjoys the following properties which characterise the construction:

- $\mathcal{L}(\text{Min}(M)) = \mathcal{L}(M)$ .
- If  $\mathcal{L}(M') = \mathcal{L}(M)$  and  $|M'| \leq |\text{Min}(M)|$  then  $M' \cong \text{Min}(M)$ .

Here  $|M|$  is the number of states of the DFA  $M$ , and  $\cong$  means the two DFAs are **isomorphic**: that is, identical apart from a possible renaming of states.

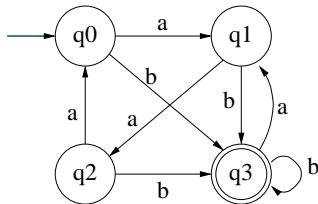
Two consequences of the above are:

- $\text{Min}(M) \cong \text{Min}(M')$  **if and only if**  $\mathcal{L}(M) = \mathcal{L}(M')$ .
- $\text{Min}(\text{Min}(M)) \cong \text{Min}(M)$ .

For a formal treatment of minimization, see Kozen chapters 13–16.

## Challenge question

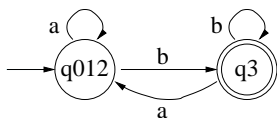
Consider the following DFA over  $\{a, b\}$ .



How many states does the minimized DFA have?

# Solution

The minimized DFA has just **2 states**:



The minimized DFA has been obtained by squishing together states  $q_0$ ,  $q_1$  and  $q_2$ . Clearly  $q_3$  must be kept distinct.

Note that the corresponding language consists of **all strings ending with b**.

## Minimization in practice

Let's look again at our definition of **equivalent states**:

*states  $q, q'$  such that: every string accepted starting from  $q$  is accepted starting from  $q'$ , and vice versa.*

This is fine as an abstract **mathematical** definition of equivalence, but it doesn't seem to give us a way to **compute** which states are equivalent: we'd have to 'check' infinitely many strings  $x \in \Sigma^*$ .

Fortunately, there's an actual **algorithm** for DFA minimization that works in reasonable time.

This is useful in practice: we can specify our DFA in the most convenient way without worrying about its size, then minimize to a more 'compact' DFA to be implemented e.g. in hardware.

# An algorithm for minimization

First eliminate any unreachable states (easy).

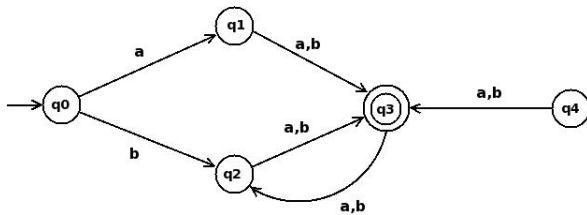
Then create a table of all possible pairs of states  $(p, q)$ , initially **unmarked**. (E.g. a two-dimensional array of booleans, initially set to false.) We **mark** pairs  $(p, q)$  as and when we discover that  $p$  and  $q$  **cannot** be equivalent.

- 1 Start by marking all pairs  $(p, q)$  where  $p \in F$  and  $q \notin F$ , or vice versa.
- 2 Look for unmarked pairs  $(p, q)$  such that for some  $u \in \Sigma$ , the pair  $(\delta(p, u), \delta(q, u))$  is marked. Then mark  $(p, q)$ .
- 3 Repeat step 2 until no such unmarked pairs remain.

If  $(p, q)$  is still unmarked, can collapse  $p, q$  to a single state.

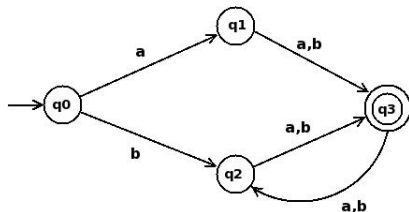
## Illustration of minimization algorithm

Consider the following DFA over  $\{a, b\}$ .



# Illustration of minimization algorithm

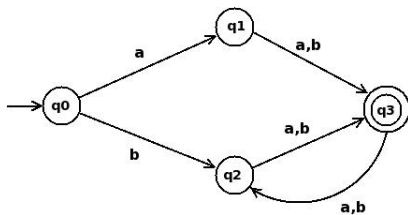
After eliminating unreachable states:





# Illustration of minimization algorithm

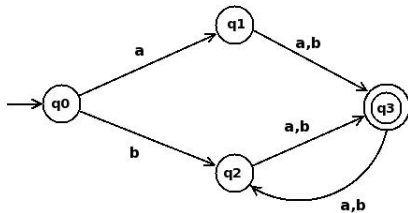
We mark states to be kept distinct using a half matrix:



q0				
q1	·			
q2	·	·		
q3	·	·	·	
		q0	q1	q2
		q3		

# Illustration of minimization algorithm

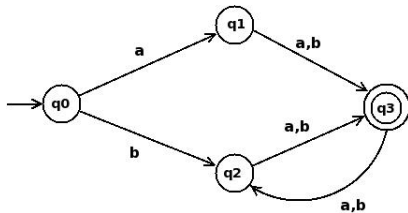
First mark accepting/non-accepting pairs:



q0				
q1	.			
q2	.	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

# Illustration of minimization algorithm

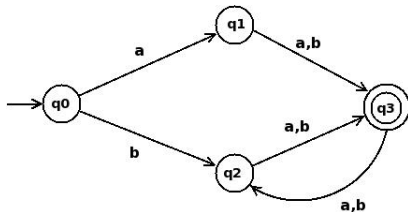
$(q_0, q_1)$  is unmarked,  $q_0 \xrightarrow{a} q_1$ ,  $q_1 \xrightarrow{a} q_3$ , and  $(q_1, q_3)$  is marked.



q0				
q1	.			
q2	.	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

# Illustration of minimization algorithm

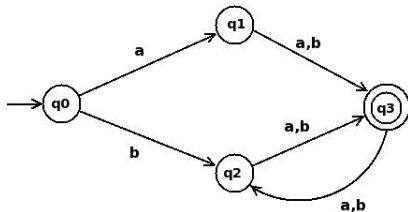
So mark  $(q_0, q_1)$ .



q0				
q1	✓			
q2	.	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

# Illustration of minimization algorithm

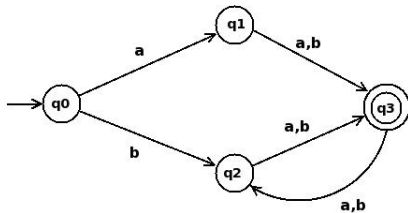
$(q_0, q_2)$  is unmarked,  $q_0 \xrightarrow{a} q_1$ ,  $q_2 \xrightarrow{a} q_3$ , and  $(q_1, q_3)$  is marked.



q0				
q1	✓			
q2	.	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

# Illustration of minimization algorithm

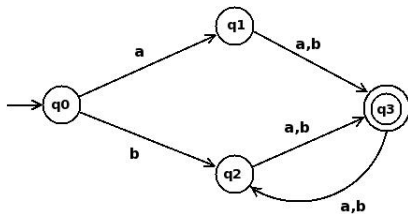
So mark  $(q_0, q_2)$ .



q0				
q1	✓			
q2	✓	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

# Illustration of minimization algorithm

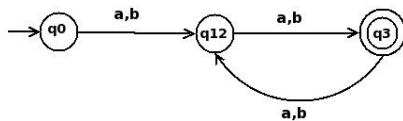
The only remaining unmarked pair  $(q_1, q_2)$  stays unmarked.



q0				
q1	✓			
q2	✓	.		
q3	✓	✓	✓	
	q0	q1	q2	q3

## Illustration of minimization algorithm

So obtain mimized DFA by collapsing  $q_1, q_2$  to a single state.





## Improving determinization

Now we have a minimization algorithm, the following improved determinization procedure is possible.

To determinize an NFA  $M$  with  $n$  states:

- 1 Perform the subset construction on  $M$  to produce an equivalent DFA  $N$  with  $2^n$  states.
- 2 Perform the minimization algorithm on  $N$  to produce a DFA  $\text{Min}(N)$  with  $\leq 2^n$  states.

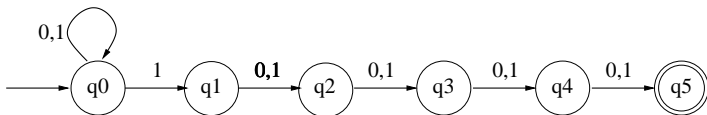
Using this method we are guaranteed to produce the smallest possible DFA equivalent to  $M$ .

In many cases this avoids the exponential state-space blow-up.

In some cases, however, an exponential blow-up is unavoidable.

## End-of-last-lecture challenge question

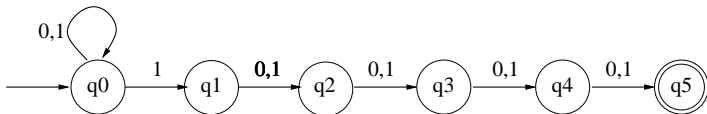
Consider last lecture's example NFA over  $\{0, 1\}$ :



What is the number of states of the smallest DFA that recognises the same language?

## End-of-last-lecture challenge question

Consider last lecture's example NFA over  $\{0, 1\}$ :

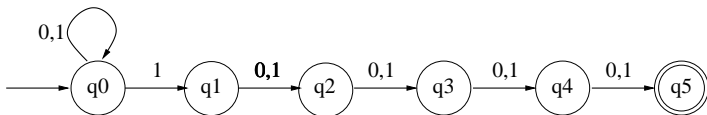


What is the number of states of the smallest DFA that recognises the same language?

**Answer:** The smallest DFA has 32 states.

## End-of-last-lecture challenge question

Consider last lecture's example NFA over  $\{0, 1\}$ :



What is the number of states of the smallest DFA that recognises the same language?

**Answer:** The smallest DFA has 32 states.

More generally, the smallest DFA for the language:

$$\{x \in \Sigma^* \mid \text{the } n\text{-th symbol from the end of } x \text{ is } 1\}$$

has  $2^n$  states. Whereas, there is an NFA with  $n + 1$  states.

## End-of-lecture questions

- ① Show that the construction of a complement automaton (used to prove that regular languages are closed under complementation) does not work for nondeterministic finite automata.

Specifically, find an example of an NFA  $M = (Q, \Delta, S, F)$  for which the NFA  $(Q, \Delta, S, Q - F)$ , obtained by swapping accepting and non-accepting states, does not recognize  $\Sigma^* - \mathcal{L}(M)$ .

**Answer:** For example, over the alphabet  $\Sigma = \{a\}$  consider

$$(\{q_0, q_1\}, \{(q_0, a, q_0), (q_1, a, q_1)\}, \{q_0, q_1\}, \{q_0\})$$

Here  $\{q_0, q_1\}$  is the set of start states, and  $\{q_0\}$  is the set of accepting states. This recognises the full language  $\Sigma^*$ . If accepting and non-accepting states are swapped, the new set of accepting states is  $\{q_1\}$ , and the language recognised is still  $\Sigma^*$ .

## End-of-lecture questions

- 2 Suppose  $N_1 = (Q_1, \Delta_1, S_1, F_1)$  is an DFA for  $L_1$ , and  $N_2 = (Q_2, \Delta_2, S_2, F_2)$  is an DFA for  $L_2$ .  
Give a direct construction of a DFA  $N$  such that  $\mathcal{L}(N) = \mathcal{L}(N_1) \cup \mathcal{L}(N_2)$ .

**Answer:** Use the identity:

$$L_1 \cup L_2 = \Sigma^* - ((\Sigma^* - L_1) \cap (\Sigma^* - L_2))$$

to reduce the construction to a composite of complementation and intersection constructions.

More directly (but equivalently), simply modify the product automaton construction on slide 5 with

$$(q, r) \xrightarrow{a} (q', r') \in \Delta' \iff q \xrightarrow{a} q' \in \Delta_1 \text{ or } r \xrightarrow{a} r' \in \Delta_2$$

# Reading

## Relevant reading:

- Closure properties of regular languages: Kozen chapter 4.
- Minimization: Kozen chapters 13–14.

## Next time:

- Regular expressions and Kleene's Theorem.  
(Kozen chapters 7–9.)