The pumping lemma

Informatics 2A: Lecture 8

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6 October, 2011

1. Showing a language isn't regular

2. The pumping lemma

3. Applying the pumping lemma

Non-regular languages

We have hinted before that not all languages are regular. E.g.

- The language $\{a^n b^n \mid n \geq 0\}$.
- The language of all *well-matched* sequences of brackets (,).
  N.B. A sequence x is well-matched if no initial subsequence y of x contains more ')' than '('.

But how do we know these languages aren't regular?

And can we come up with a general technique for proving the non-regularity of languages?

The basic intuition: DFAs can't count!

Consider $L = \{a^n b^n \mid n \geq 0\}$. Just suppose, hypothetically, there *were* some DFA M with $\mathcal{L}(M) = L$.

Suppose furthermore that M had just processed a^n, and some continuation b^m was to follow.

Intuition: M would need to have *counted* the number of a's, in order to know how many b's to expect.

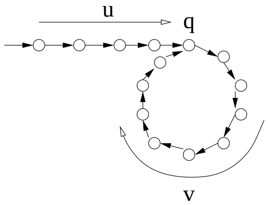
More precisely, let q_n denote the state of M after processing a^n. Then for any $m \neq n$, the states q_m, q_n must be different, since b^m takes us to an accepting state from q_m, but not from q_n.

In other words, M would need infinitely many states, one for each natural number. Contradiction!

Put slightly differently...

Suppose there were some DFA M for $L = \{a^n b^n \mid n \geq 0\}$. Then M would have some finite number of states, say k.

Now consider what happens when we feed M with the string a^k. It passes through a sequence of $k + 1$ states (including the initial state). So there *must* be some state q that's visited twice or more:



This means the string a^k can be decomposed as uvw, where

- u takes M from the initial state to q,
- v takes M once round the loop from q to q,
- w is whatever is left of a^k after uv.

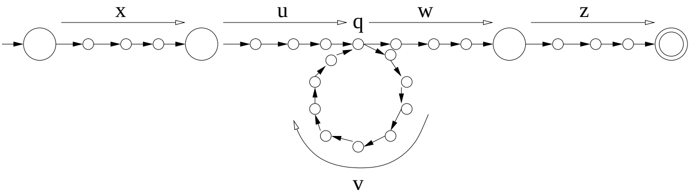
(Note that u and w might be ϵ, but v definitely isn't.)

More generally. . .

If L is *any* regular language, we can pick *some* corresponding DFA M, and it will have some number of states, say k.

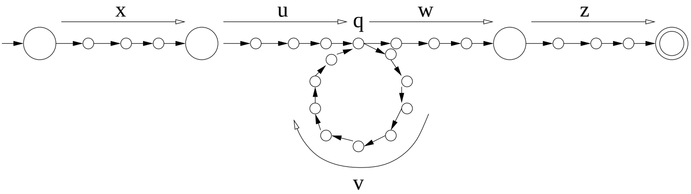
Not only must every string of length $\geq k$ cause a revisited state — so must every substring of length $\geq k$ *within* such a string.

Indeed, consider what happens when we run M on a string $xyz \in L$, where $|\ y\ | \geq k$. There must be at least one state q we visit twice in the course of processing y:



(There may be other 'revisited states' not indicated here.)

The idea of 'pumping'



So y can be decomposed as uvw, where

- xu takes M from the initial state to q,
- $v \neq \epsilon$ takes M once round the loop from q to q,
- wz takes M from q to an accepting state.

But now M will be oblivious to whether, or how many times, we go round the v-loop!

So we can 'pump in' as many copies of the substring v as we like, knowing that we'll still end in an accepting state.

The pumping lemma: official form

The pumping lemma basically summarizes what we've just said.

Pumping Lemma. Suppose L is a regular language. Then L has the following property.

> *(P) There exists* $k \geq 0$ *such that for any strings* x, y, z *with* $xyz \in L$ *and* $\mid y \mid \geq k$*, there exist strings* u, v, w *such that* $y = uvw$*,* $v \neq \epsilon$*, and for every* $i \geq 0$ *we have* $xuv^iwz \in L$.

The pumping lemma: contrapositive form

Since we want to use the pumping lemma to show a language *isn't* regular, we usually apply it in the following equivalent but back-to-front form.

Suppose L is a language for which the following property holds:

> ($\neg P$) *For all* $k \geq 0$ *such that there exist strings* x, y, z *with* $xyz \in L$ *and* $|\, y \,| \geq k$, *and for all* u, v, w *such that* $y = uvw$ *and* $v \neq \epsilon$, *there exists* $i \geq 0$ *we have* $xuv^i wz \notin L$.

Then L is not a regular language.

N.B. The pumping lemma can only be used to show a language *isn't* regular. Showing L satisfies (P) doesn't prove L is regular!

To show some language *is* regular, give a DFA or NFA or regular expression that defines it.

The pumping lemma: a user's guide

So to show some language L is not regular, it's enough to show that L satisfies $\neg$P.

Note that $\neg$P is quite a complex statement: $\forall \cdots \exists \cdots \forall \cdots \exists \cdots$.

It's helpful to think in terms of how you would refute an opponent who claimed to have a DFA for L.

We'll look a simple example first, then offer some advice on the general pattern of argument.

Example 1

Consider $L = \{a^n b^n \mid n \geq 0\}$.

Suppose there were some DFA for L, and it had k states.
(k is chosen by 'opponent' — we just have to cope.)

Consider the strings $x = \epsilon$, $y = a^k$, $z = b^k$. Note that $xyz \in L$ and $\mid y \mid \geq k$ as required.
(y is cunningly chosen by 'us'.)

Suppose now we're given a decomposition of y as uvw with $v \neq \epsilon$.
(u, v, w chosen by 'opponent' — we have to cope.)

Let $i = 0$ Then $uv^i w = uw = a^l$ for some $l < k$. So $xuv^i wz = a^l b^k \notin L$, and we win!
(i chosen by 'us'.)

Thus L satisfies $\neg$P, so L isn't regular.

Use of pumping lemma: general pattern

- The opponent proposes a 'number of states' k. (That is, he claims he has a DFA for L, and tells you its number of states.) You don't get to choose k — you have to cope with what the opponent throws at you.
- You respond with a cunning choice of strings x, y, z, which might depend on k. These must satisfy $xyz \in L$ and $\mid y \mid \geq k$. Also, y should be chosen to 'disallow pumping' ...
- The opponent picks a decomposition of y as uvw with $v \neq \epsilon$. Again, you just have to cope with his choice.
- Finally, you have to choose i $(\neq 1)$ such that $xuv^iwz \notin L$. Here i might depend on all the previous data.

Example 2

Consider $L = \{a^{n^2} \mid n \geq 0\}$.

Suppose there were a DFA for L with k states.

Let $x = a^{k^2-k}$, $y = a^k$, $z = \epsilon$, so $xyz = a^{k^2} \in L$.

Given any splitting of y as uvw with $v \neq \epsilon$, we have $1 \leq \mid v \mid \leq k$.

So taking $i = 2$, we have $xuv^2wz = a^n$ where $k^2 + 1 \leq n \leq k^2 + k$.

But there are no perfect squares between k^2 and $k^2 + 2k + 1$, so n isn't a perfect square. Thus $xuv^2wz \notin L$.

By the pumping lemma, we conclude that L is not regular.

Three clicker questions

For each of the following languages over $\{a, b\}$, decide whether they are regular or not.

Press 1 for regular, 2 for non-regular.

- Strings with an odd number of a's and an even number of b's.
- Strings containing more a's than b's.
- Strings such that (no. of a's) * (no. of b's) $\equiv$ 6 (mod. 24)

Reading and prospectus

Relevant reading: Kozen chapters 11, 12.

That concludes the part of the course on regular languages.
In some informal sense, you now know 'everything' about the theory of regular languages.

Next time, we start on the next level up in the Chomsky hierarchy: context-free languages.