On Soddy's Hexlet and a Linked 4-Pair

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Abstract. An n-cycle of balls is a cyclic sequence of non-overlapping n balls in R^3 in which each consecutive pair of balls are tangent. An (m, n)-link is a pair of an m-cycle and an n-cycle that form a non-splittable link, with no two balls overlapping. It is proved that (1) a $(3, n)$-link exists only when $n \geq 6$, and in any $(3, 6)$-link, each ball in the 3-cycle is tangent to all balls in the 6-cycle, (2) a $(4, 4)$-cycle exists and in any $(4, 4)$-cycle, each ball in a 4-cycle is tangent to all balls in the other 4-cycle.

1 Introduction

A *cycle* of n balls (simply an n-*cycle*) is a cyclic sequence of non-overlapping balls $B_1 B_2 \ldots B_n$ in R^3 such that for each $i = 1, 2, \ldots, n$, B_i is tangent to B_{i+1}, where $B_{n+1} = B_1$. The sizes of the balls may be different. The *string* of a cycle is the closed polygonal curve consisting of the line-segments each connecting the centers of a pair of consecutive balls in the cycle. A cycle is called *knotted* if its string forms a non-trivial knot. In [2], it was proved, under the restriction that only unit balls lying between a pair of parallel planes distance $2 + \sqrt{2}$ apart are available, that 16 balls are necessary and sufficient to make a knotted cycle (a trefoil knot). For elementary facts on knots and links, see Adams [1].

When we consider a pair of cycles, then we always assume that balls in distinct cycles never overlap. By an (m, n)-*link*, we mean a pair of m-cycle and n-cycle whose strings form a non-splittable link. In this paper, we prove the following two fundamental theorems. These results are applied in [3].

Theorem 1. *A* $(3, n)$*-link exists if and only if* $n \geq 6$*, and in any* $(3, 6)$*-link, each ball in the* 6*-cycle is tangent to all balls in the* 3*-cycle.*

If each ball in a 6-cycle is tangent to all balls in a 3-cycle, then the 6-cycle is called a Soddy's hexlet (with respect to the 3-cycle). Frederick Soddy discovered such a configuration of balls in 1936, see [4]. Thus, the 6-cycle in a $(3, 6)$-link is a Soddy's hexlet.

A pair of 4-cycles is called a *linked* 4*-pair* if they form a (4,4)-link.

Theorem 2. *A linked* 4*-pair exists, and in any linked* 4*-pair, each ball in a* 4*-cycle is tangent to all balls in the other* 4*-cycle.*

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2 Two Examples

Example 1 Let A_1 be a unit ball inscribed between the planes $z = \pm 1$. Let $B_1, B_2, \ldots B_6$ be unit balls inscribed between the planes $z = \pm 1$, girdling the ball A_1. Let A_2 be the half-space $z \geq 1$ and A_3 be the half-space $z \leq -1$. Then by an inversion of R^3 with respect to some point on the xy-plane, $A_1A_2A_3$ and $B_1B_2 \ldots B_6$ are transformed into a $(3, 6)$-link.

Example 2 Let B_1 be the half-space $z \geq 1$, and B_4 be the half-space $z \leq -1$ in R^3, and let $A_1, A_2, A_3, A_4;\ B_2, B_3$ be the balls listed in the following table:

ball	center	radius
A_1	$(\sqrt{2}, 0, 0)$	1
A_2	$(0, \sqrt{2}, 0)$	1
A_3	$(-\sqrt{2}, 0, 0)$	1
A_4	$(0, -\sqrt{2}, 0)$	1
B_2	$(0, 0, 1/2)$	$1/2$
B_3	$(0, 0, -1/2)$	$1/2$

Then, no two of A_i, B_j $(i = 1, 2, 3, 4)$ overlap, and $A_1A_2A_3A_4$ is a 4-cycle, and B_i is tangent to B_{i+1} for $i = 1, 2, 3$; the two half-space B_1, B_4 are disjoint. It is not difficult to see that by an inversion φ of R^3 with respect to a point, say, $(3, 0, 0)$, we can get a linked 4-pair

$$\varphi(A_1)\varphi(A_2)\varphi(A_3)\varphi(A_4) \text{ and } \varphi(B_1)\varphi(B_2)\varphi(B_3)\varphi(B_4).$$

3 Proof of Theorem 1

The existence of a (3,6)-link is shown in Example 1.

Let $A_1A_2A_3$ and $B_1B_2 \ldots B_n$ be a 3-cycle and an n-cycle that form together a (3,n)-link. Let p be the contact point of A_2, A_3, and φ be an inversion of R^3 with respect to the point p. Denote the boundary of A_i by ∂A_i. Then $H_i := \varphi(\partial A_i)$, $i = 2, 3$ are a pair of parallel planes, and $\varphi(A_1)$ is a ball tangent to these two planes. We may suppose that $\varphi(A_1)$ is a unit ball with center $(0, 0, 1)$ and H_2 is the xy-plane, H_3 is the plane $z = 2$. The n balls $\varphi(B_j)$, $j = 1, \ldots, n$, lie between the planes H_1, H_2. Since $A_1A_2A_3$, $B_1B_2 \ldots B_n$ form a (3,n)-link, the balls $\varphi(B_j)$, $j = 1, \ldots, n$, must surround the ball $\varphi(A_1)$. Let (x_j, y_j, z_j) be the center of $\varphi(B_j)$ and let $p_j = (x_j, y_j, 0)$. Since the n balls $\varphi(B_j)$ surround the ball $\varphi(A_1)$, we must have

$$\sum_{j=1}^{n} \angle p_j O p_{j+1} \geq 2\pi,$$

where $O = (0, 0, 0)$ and $p_{n+1} = p_1$. Let r_j be the radius of $\varphi(B_j)$. Then

$$\begin{aligned} \|p_j - p_{j+1}\|^2 &= (r_j + r_{j+1})^2 - (z_j - z_{j+1})^2 \\ &= (r_j + r_{j+1} + z_j - z_{j+1})(r_j + r_{j+1} - z_j + z_{j+1}). \end{aligned}$$

Since $r_{j+1} \leq z_{j+1}$ and $r_{j+1} + z_{j+1} \leq 2$, we have

$$\|p_j - p_{j+1}\|^2 \leq (r_j + z_j)(2 + r_j - z_j).$$

On the other hand,

$$\|p_j - O\|^2 \geq (r_j + 1)^2 - (z_j - 1)^2 = (r_j + z_j)(2 + r_j - z_j).$$

Hence $\|p_j - p_{j+1}\| \leq \|p_j - O\|$. Similarly, we have $\|p_j - p_{j+1}\| \leq \|p_{j+1} - O\|$. Hence, in the triangle $p_j p_{j+1} O$, the edge $p_j p_{j+1}$ is the shortest edge. Therefore $\angle p_j O p_{j+1} \leq \pi/3$. Thus, in order to have $\sum_{j=1}^{n} \angle p_j O p_{j+1} \geq 2\pi$, we must have $n \geq 6$. If $n = 6$, then it is necessary that $\angle p_j O p_{j+1} = \pi/3$ for all $j = 1, \ldots, 6$, that is $\|p_j - p_{j+1}\| = \|p_j - O\|$ for all $j = 1, \ldots, 6$. Hence, we have $r_{j+1} = z_{j+1}$ and $r_{j+1} + z_{j+1} = 2$ for all j. Therefore, all $\varphi(B_j)$ are tangent to $H_2, H_3, \varphi(A_1)$.

4 Proof of Theorem 2

Lemma 3. *Let $p_1 p_2 p_3 p_4$ be a convex quadrilateral, and D_1, D_2 be two disks in the plane such that*
(1) $p_1, p_3 \in D_1$ and $\{p_2, p_4\} \cap int(D_1) = \emptyset$,
(2) $p_2, p_4 \in D_2$ and $\{p_1, p_3\} \cap int(D_2) = \emptyset$,
where $int(D_i)$ denotes the interior of D_i. Then D_1 and D_2 are the same disk, and all p_1, p_2, p_3, p_4 lie on the boundary of D_1.

Proof. It follows from (1) that $\angle p_2 + \angle p_4 \leq 180°$, with the equality only when all p_i lie on the boundary of D_1. From (2), $\angle p_1 + \angle p_3 \leq 180°$ with the equality only when all p_i lie on the boundary of D_2. Since $\angle p_1 + \angle p_2 + \angle p_3 + \angle p_4 = 360°$, we must have $\angle p_1 + \angle p_3 = 180°$ and $\angle p_2 + \angle p_4 = 180°$. Hence p_1, p_2, p_3, p_4 lie on the boundary of D_1, and also lie on the boundary of D_2. Therefore D_1 and D_2 are the same disk. □

For two disjoint balls B_1, B_3 in R^3 with centers b_1, b_3, respectively, let $\Omega(B_1, B_3)$ denote the locus of the center of a ball that is tangent externally to both B_1, B_3. It is clear that if B_1, B_3 are of the same size, then $\Omega(B_1, B_3)$ is the plane perpendicularly bisecting the line-segment $\overline{b_1 b_3}$. What is $\Omega(B_1, B_3)$ if the sizes of B_1, B_3 are different? Let H be a plane that contains the line $b_1 b_3$. Then, it is not difficult to see that $\Omega(B_1, B_3) \cap H$ is a branch of a hyperbola in H with foci b_1, b_3. Hence $\Omega(B_1, B_3)$ is the surface obtained by rotating this curve around the line $b_1 b_3$, that is, one sheet of a 2-sheet-hyperboloid of revolution with the foci b_1, b_3. Hence the next lemma is clear.

Lemma 4. *Let B_1, B_3 be disjoint balls in R^3 with centers b_1, b_3, respectively. Then for any point $p \in R^3$, the polygonal curve $b_1 p\, b_3 := \overline{b_1 p} \cup \overline{p\, b_3}$ intersects $\Omega(B_1, B_3)$ only in one point.* □

Proof of Theorem 2. The existence of a linked 4-pair is shown in Example 2. To prove the latter part of Theorem 2, let $A_0A_1A_2A_3$, $B_0B_1B_2B_3$ be a pair of linked 4-cycles.

First, suppose that A_0 is tangent to B_0. Let b_i be the center of B_i for $i = 0, 1, 2, 3$. By inverting R^3 with respect to the contact point of A_0, B_0, we may suppose that A_0 is the half-space $z \leq -1$ and B_0 is the half-space $z \geq 1$. Then, all other balls lie between the planes $z = \pm 1$; A_1, A_3 are tangent to $z = -1$, B_1, B_3 are tangent to $z = 1$. For $i = 1, 3$, let c_i be the contact point of B_i and the 'ceiling' $z = 1$, v_i be the contact point of B_i and B_2 ; f_i be the contact point of A_i and the 'floor' $z = -1$, u_i be the contact point of A_i and A_2 , respectively. Since $A_0A_1A_2A_3$ and $B_0B_1B_2B_3$ form a non-splittable link, the pair of closed polygonal curves $c_1v_1v_3c_3$ and $f_1u_1u_3f_3$ form a non-splittable link. Let us use the following notations:

$b_2\!\uparrow$: the ray from b_2 that meets the ceiling $z = 1$ perpendicularly.
$b_2\!\downarrow$: the ray from b_2 that meets the floor $z = -1$ perpendicularly,
$b_2\!\updownarrow$: the line $b_2\!\uparrow \cup\, b_2\!\downarrow$.

For each $p \in b_2\!\uparrow$, the polygonal curve $b_1 p\, b_3$ intersects $\Omega(B_1, B_3)$ at a unique point $q = q(p)$ by Lemma 2. Let $Q(p)$ be the ball with center q and tangent to both B_1, B_3. Then,

$$Q(p) \cap int(A_1) = Q(p) \cap int(A_3) = \emptyset \text{ for all } p \in b_2\!\uparrow . \qquad (1)$$

This is proved later. For $i = 1, 3$, let $w_i = w_i(p)$ be the contact point of $Q(p), B_i$. If p moves continuously, then the line w_1w_3 moves continuously. If $p = b_2$, then $w_i = v_i$, $i = 1, 3$, and if the altitude (= z-coordinate) of p tends to infinity, then w_i tends to c_i. Thus the closed polygonal curve $c_1w_1w_3c_3$ changes its shape as p climbs up $b_2\!\uparrow$ and the pair $(f_1u_1u_3f_3, c_1w_1w_3c_3)$ changes from a non-splittable link to a splittable link. Hence, at some point $p = p_0 \in b_2\!\uparrow$, the two curves must intersect. Then, since $Q(p)$ cannot overlap A_1, A_3, the intersection must occur between the line-segments $\overline{w_1w_3}$ and $\overline{u_1u_3}$. Hence, at $p = p_0$, $w_1u_1w_3u_3$ forms a convex quadrilateral lying on a plane. Then, by applying Lemma 1, it follows that the sections of $Q(p_0), A_2$ by this plane coincide. Therefore A_2 is tangent to 4 balls A_1, A_3, B_1, B_3. Similarly, it follows that B_2 is tangent to 4 balls A_1, A_3, B_1, B_3. Thus, if A_0, B_0 are tangent to each other, then A_1 is tangent to B_2, and A_2 is tangent to B_1.

Let us write $A_i \top B_j$ if A_i and B_j are tangent to each other. Then the above argument proves that $A_i \top B_j \implies A_{i+1} \top B_{j+2},\ A_{i+2} \top B_{j+1},$ where the indexes are taken mod 4. Therefore,

$$\begin{aligned} A_i \top B_j &\Rightarrow A_{i+1} \top B_{j+2} \Rightarrow A_{i+2} \top B_{j+4} \Rightarrow A_{i+4} \top B_{j+5} \\ A_i \top B_j &\Rightarrow A_{i+2} \top B_{j+1} \Rightarrow A_{i+4} \top B_{j+2} \Rightarrow A_{i+5} \top B_{j+4}, \end{aligned}$$

that is,

$$A_i \top B_j \Rightarrow A_i \top B_{j+1},\ A_{i+1} \top B_j.$$

Hence, if $A_0 \top B_0$ then $A_i \top B_j$ for any i, j. Thus, if some A_i is tangent to some B_j, then each A_i is tangent to all B_j.

Next, suppose that for any i, j, A_i is not tangent to B_j. Then we translate $A_0A_1A_2A_3$ as a rigid body, in the direction $\overrightarrow{b_0a_0}$, until some A_i touches some B_j. Once this happens, then every A_i touches all B_j simultaneously by the above argument. However, since A_0 goes away from B_0 by our translation, A_0 is never tangent to B_0, a contradiction. Hence there always exists an A_i that is tangent to some B_j, and hence each A_i is tangent to all B_j.

Now, we proceed to the proof of (1). It is enough to show that $Q(p) \cap int(A_1) = \emptyset$. We may suppose that q lies on $p\,b_1$, and $p \neq b_2$. Then $b_1 \notin b_2\!\uparrow$. (For otherwise, since $Q(p)$ is tangent to B_3, q must lie on $\overrightarrow{b_1b_2}$, and hence the ray $\overrightarrow{b_1b_2}$ from the focus b_1 intersects $\Omega(B_1, B_3)$ at two points q, b_2, a contradiction.) Let P be the ball centered at p and externally tangent to B_1. Then P contains $Q(p)$. Hence, to prove $Q(p) \cap int(A_1) = \emptyset$, it is sufficient to prove $P \cap int(A_1) = \emptyset$. Let H be the plane determined by b_1 and $b_2\!\updownarrow$, and let $H_+ \subset H$ be the half-plane bounded by $b_2\!\updownarrow$ and containing b_1. Rotate the ball A_1 around the line $b_2\!\updownarrow$ until its center comes to lie on H_+, and denote by A the ball at this position. Then, it is enough to show that $P \cap int(A) = \emptyset$. Note that $B_2 \cap int(A) = \emptyset$, and A is tangent to the floor $z = -1$, but A and B_1 may overlap each other. Let s be the south pole of B_2, that is, the point where $b_2\!\downarrow$ meets the boundary of B_2, and f be the contact point of A and the floor $z = -1$, see Figure 1. Let $\widehat{v_1s}$ denote the minor arc of $H \cap \partial B_2$. Then, since $p \neq b_2$, it follows that $P \cap \widehat{v_1s} = \emptyset$.

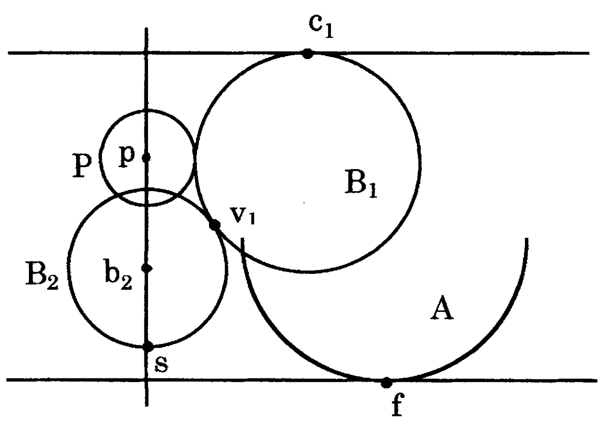

Fig. 1. The sections by the plane H

Suppose that $P \cap int(A) \neq \emptyset$, and let t be a point in $H \cap P \cap int(A)$. Notice that tf and $\widehat{v_1s} \cup c_1v_1$ intersect. Since $tf \cap \widehat{v_1s} = \emptyset$ (for otherwise, $int(A) \cap B_2 \neq \emptyset$, a contradiction), tf intersects c_1v_1, that is, tv_1fc_1 forms a convex quadrilateral in the plane H. Since $t, f \in A$, $\{c_1, v_1\} \cap int(A) = \emptyset$, and $c_1, v_1 \in B_1$, $\{t, f\} \cap int(B_1) = \emptyset$, it follows from Lemma 1 that t, v_1, f, c_1 all lie on the boundary of A, which contradicts that $t \in int(A)$. Therefore $P \cap int(A) = \emptyset$. This completes the proof of Theorem 2. $\square$

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