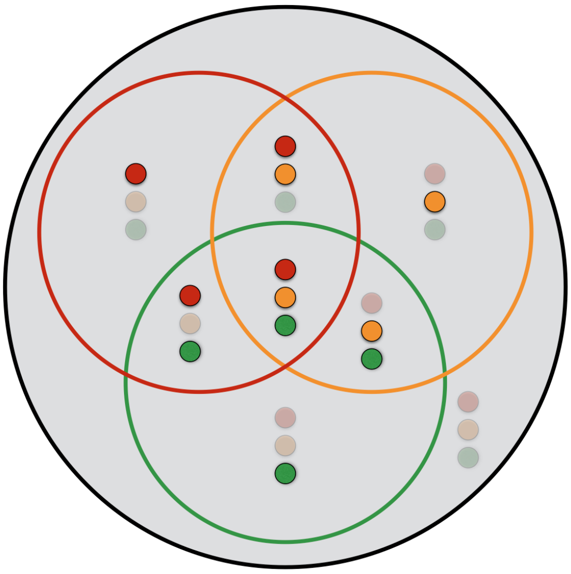
Informatics 1 - Computation & Logic: Tutorial 1

Propositional Logic: Venn Diagrams and Truth Tables

Week 3: 2-6 October 2017

Please attempt the entire worksheet in advance of the tutorial, and bring all work with you. Tutorials cannot function properly unless you study the material in advance. Attendance at tutorials is **obligatory**; please let the ITO know if you cannot attend.

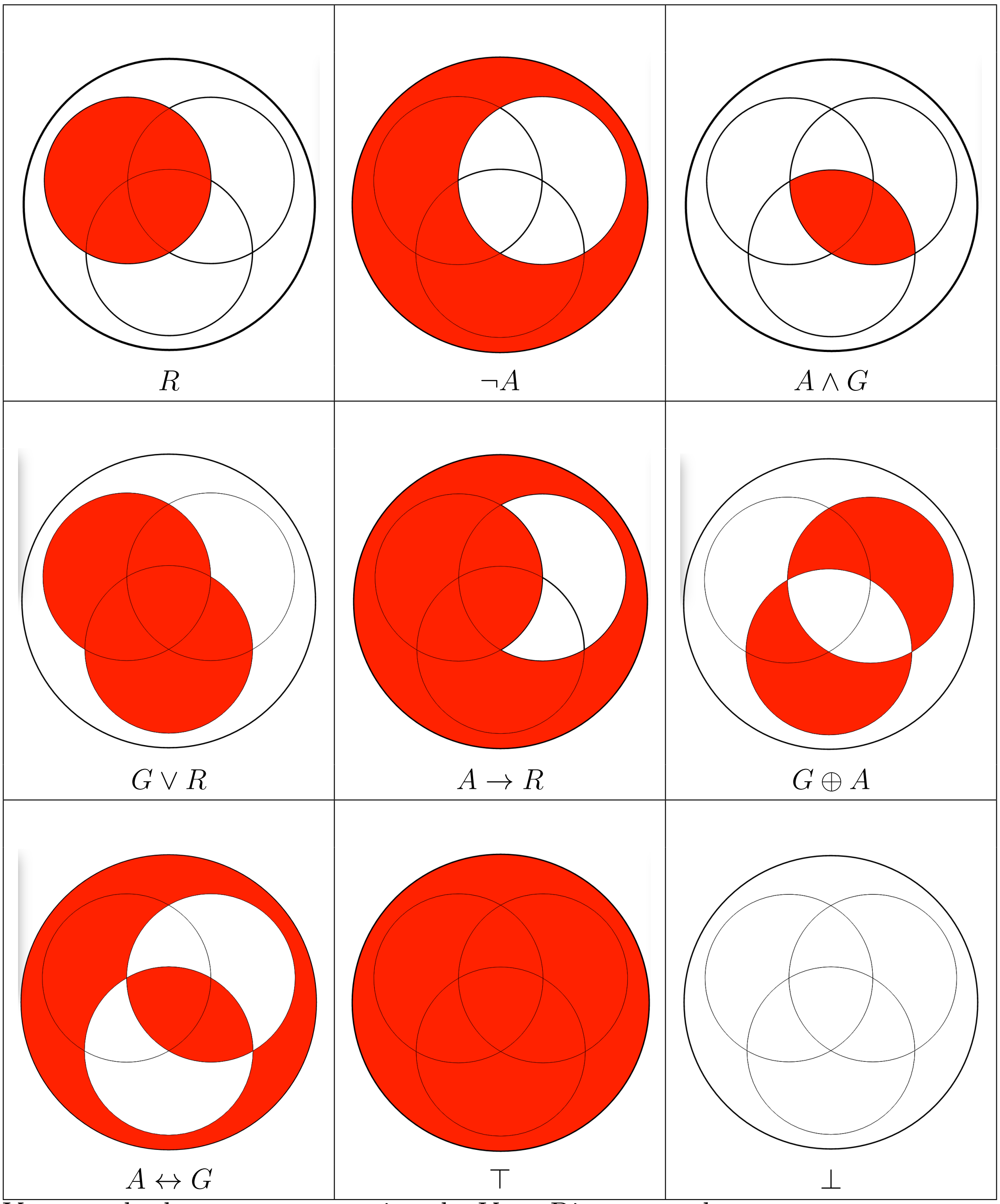
You may work with others, indeed you should do so; but you must develop your own understanding; you can't phone a friend during the exam. If you do not master the coursework you are unlikely to pass the exams.



A Venn diagram is, in essence, a visual truth table. In the blank diagrams below, each circle represents a region in which a given logical atom, is true; R, A, and G, going clockwise from top left. Where the circles overlap, two or three of the atoms are true. The diagram here represents the atoms by three lights, which may be on (true) or off (false), to show the state in each of the eight regions.

1. In this exercise, you will be asked to translate between truth tables, logical expressions, and Venn diagrams.

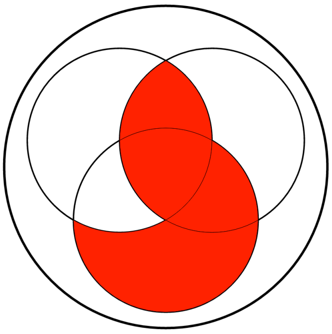
   (a) For the nine examples, shade in the appropriate regions of the Venn diagram to show the states in which the expression is true.



You can check your answers using the Venn Diagram maker at

https://www.inf.ed.ac.uk/teaching/courses/inf1/cl/tools/venn/,
but please work them out for yourself first.

(b) Next, look at these truth tables, and shade in the Venn diagram accordingly; then, see if you can figure out corresponding logical expression; note that while there is only one correct diagram for each truth table, there are infinitely many equivalent expressions. Shorter expressions are in general to be preferred here to longer ones.



R	A	G	Exp: $(R\ ?\ A : G)$
$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$
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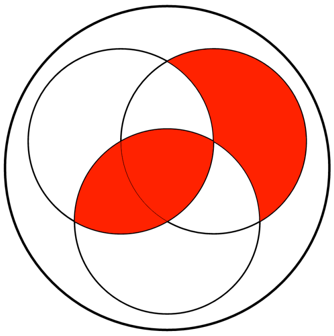

R	A	G	Exp: $(R \vee G)\ ?\ (R \wedge G) : A$
$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\bot$
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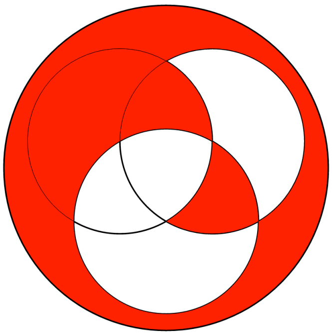

R	A	G	Exp: $G\ ?\ (A \wedge \neg R) : (A \rightarrow R)$
$\top$	$\top$	$\top$	$\bot$
$\top$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\top$
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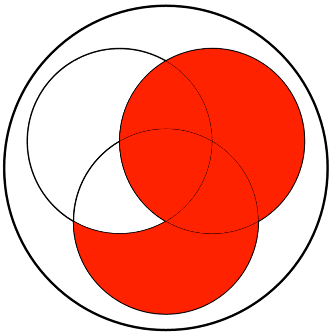
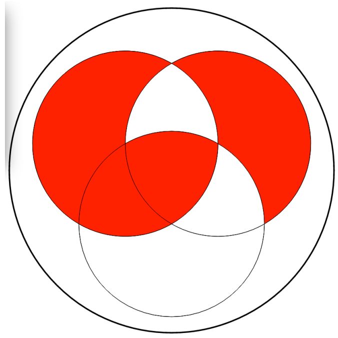


R	A	G	Exp: $G\ ?\ \neg R \vee A : A$
$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$

(c) Next, given the Venn diagram, try to complete the truth table and the expression.



R	A	G	Exp: $((R \wedge G) \vee (A \wedge \neg R \wedge \neg G)) \vee \neg(R \to A)$
$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\bot$
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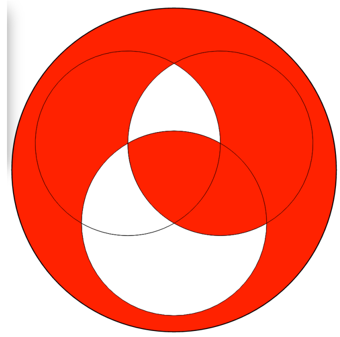
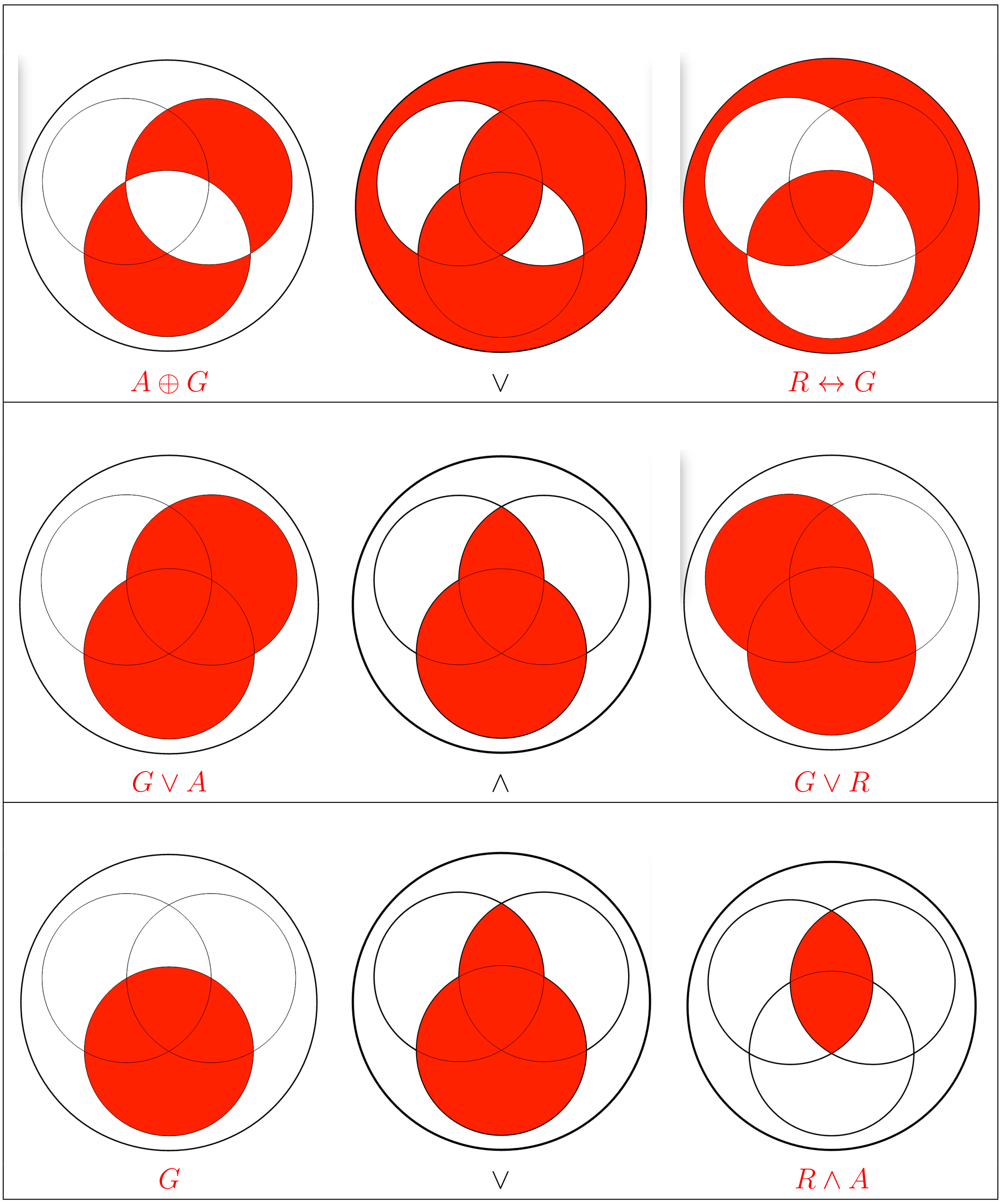


R	A	G	Exp: $(G \to A) \wedge \neg(R \wedge A \wedge \neg G)$
$\top$	$\top$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$
$\bot$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$
$\bot$	$\bot$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\top$

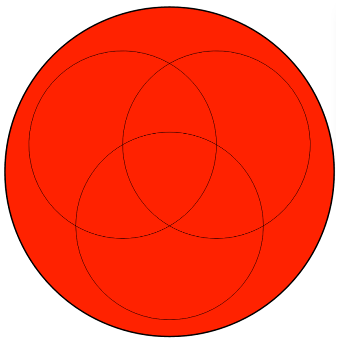
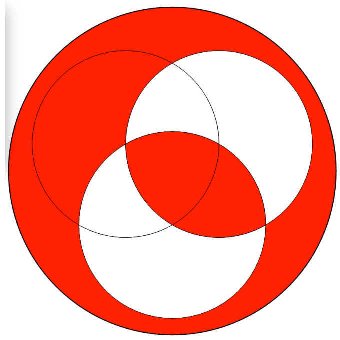
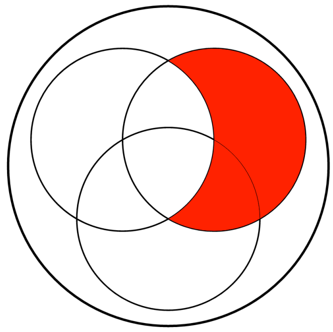
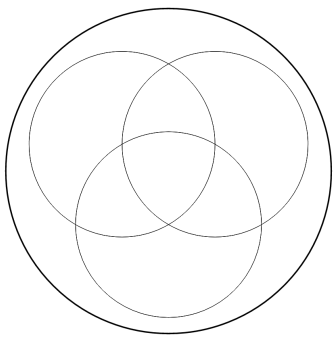
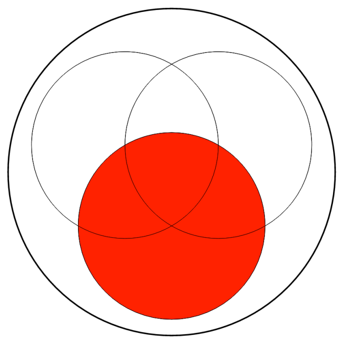
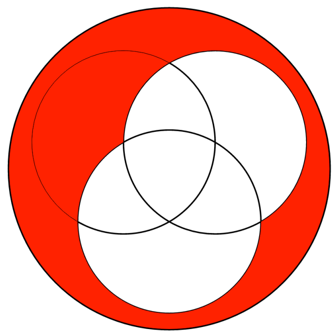
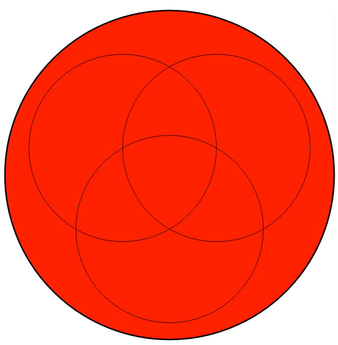
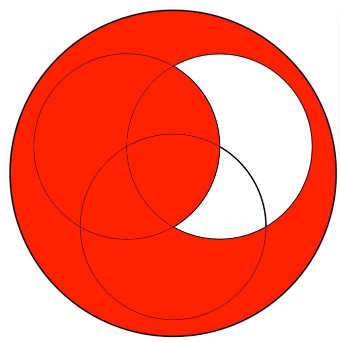
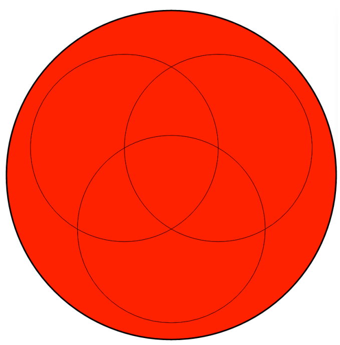
2. By now, you may be getting a feel for how Venn diagrams are combined by different operators. For this question, shade in the middle diagram to combine the left and right diagrams on the basis of the given operator. Work out the correct shading visually, then work out logical expressions and check your work using the Venn diagram generator.

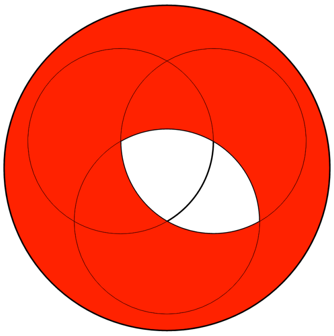


3. Now, you have probably noticed something interesting about the last two answers - well done! You've just discovered the distributivity of disjunction over conjunction! This identity is one of the laws of Boolean Algebra, which we use to reason about logical expressions.

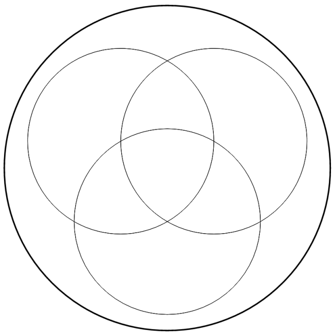
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Here are some more expressions to diagram. See how many more identities you can find, among the expressions on this question, and among the other expressions on this sheet.

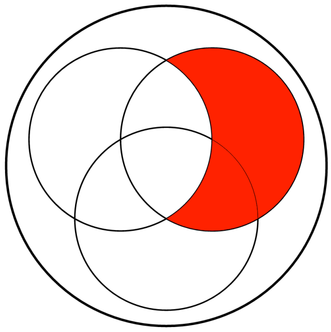
 $\neg\bot$	 $(G \to A) \wedge (A \to G)$	 $\neg(A \to R)$
 $A \wedge \neg A$	 $\neg\neg G$	 $\neg G \wedge \neg A$
 $G \vee \top$	 $\neg A \vee R$	 $A \vee \neg A$



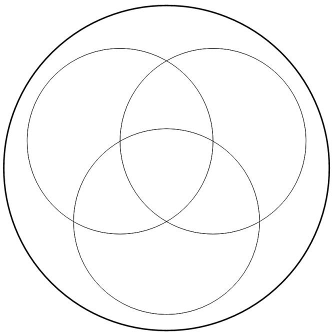
$\neg(A \wedge G)$



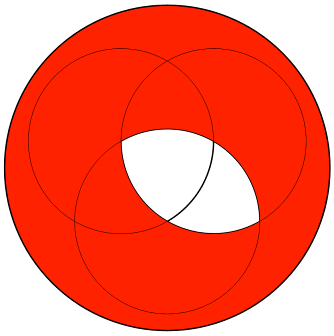
$A \wedge \bot$



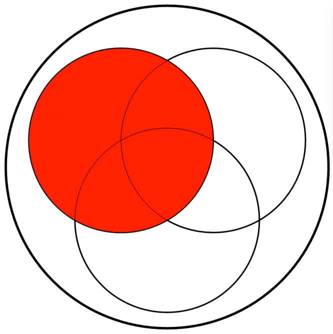
$A \wedge \neg R$



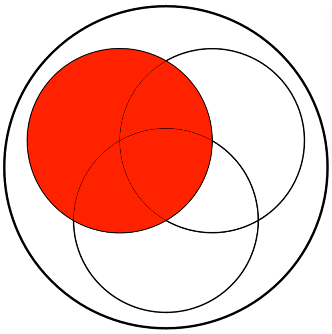
$\neg\top$



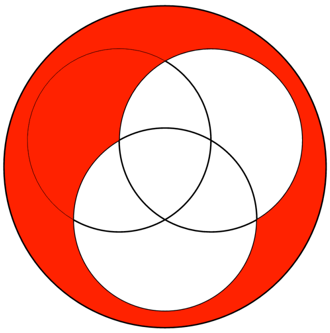
$\neg A \vee \neg G$



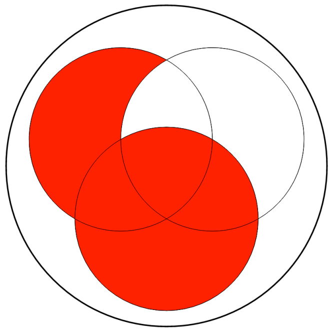
$R \vee \bot$



$R \wedge \top$



$\neg(G \vee A)$



$(R \to A) \to G$

Identities discovered:

$\neg(a \to b) = a \wedge \neg b$ $\quad$ $a \leftrightarrow b = (a \to b) \wedge (b \to a)$ $\quad$ $a \to b = \neg a \vee b$

$\neg(a \vee b) = \neg a \wedge \neg b$ $\quad$ $\neg(a \wedge b) = \neg a \vee \neg b$

$\neg 0 = 1$ $\quad$ $\neg\neg a = a$ $\quad$ $\neg 1 = 0$

$a \vee 1 = 1$ $\quad$ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\quad$ $a \wedge 0 = 0$

$a \vee 0 = a$ $\quad$ $a \vee \neg a = 1$ $\quad$ $a \wedge \neg a = 0$ $\quad$ $a \wedge 1 = a$

4. Look up the following terms (MML §6.2.4 or on the internet or elsewhere) and describe, in words, when an expression in propositional logic is:

   (a) Satisfiable:

   An expression is satisfiable IFF there exists some valuation of its atoms for which it is true

   (b) Tautologous:

   An expression is tautologous IFF it is true for all valuations of its atoms

   (c) Inconsistent:

   An expression is inconsistent IFF it is false for all valuations of its atoms

   (d) Contingent:

   An expression is contingent IFF it is satisfiable but not tautologous; that is, it is true for some valuations but not all.

5. Construct truth tables for the following expressions of propositional logic, and use these to decide whether the expressions are satisfiable, tautologous, contingent, or inconsistent:

   (a) $(R \rightarrow A) \vee (\neg A \vee \neg R)$

   Draw the truth table here:

R	A	$\neg R$	$\neg A$	$R \rightarrow A$	$(\neg R \vee \neg A)$	EXP
$\top$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$

   This expression is
   <u>SATISFIABLE</u>/<u>TAUTOLOGOUS</u>/CONTINGENT/INCONSISTENT

(b) $R \to (A \land (R \lor A))$

Draw the truth table here:

R	A	$R \lor A$	$(A \land (R \lor A))$	EXP
$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$

This expression is
<u>SATISFIABLE</u>/TAUTOLOGOUS/<u>CONTINGENT</u>/INCONSISTENT

(c) $(\neg R \land A) \lor G \oplus ((R \lor \neg A) \to G)$

Draw the truth table here:

R	A	G	$\neg R$	$\neg A$	$\neg R \land A$	$R \lor \neg A$	$(\neg R \land A) \lor G$	$((R \lor \neg A) \to G)$	EXP
$\top$	$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$

This expression is
SATISFIABLE/TAUTOLOGOUS/CONTINGENT/<u>INCONSISTENT</u>

Tutorial Activities

1. First take 15 minutes to compare answers to questions 1-5 with a buddy, then check that your group agrees on the answers.

   Ask one of the tutors if you have questions.

2. The main activity for this tutorial is to introduce you to a kind of reasoning, taking you from particular examples to a more general understanding, that is common throughout informatics.

   In lectures we have shown that $\oplus$ and $\leftrightarrow$ are associative and commutative, and that $A \oplus B \oplus C = A \leftrightarrow B \leftrightarrow C$ is true iff the string abc has odd parity[1], where $a = 1$ iff $A = \top$, and so on.
   In this question you will explore expressions such as:

   $$A \oplus B \oplus C \oplus \cdots \oplus X \oplus Y \oplus Z \quad \text{and} \quad A \leftrightarrow B \leftrightarrow C \leftrightarrow \cdots \leftrightarrow X \leftrightarrow Y \leftrightarrow Z$$

   (a) What can you say about the case with four boolean variables?

   $$A \oplus B \oplus C \oplus D \quad \text{and} \quad A \leftrightarrow B \leftrightarrow C \leftrightarrow D$$

   When are these formulae true?
   $(A \oplus B \oplus C) \oplus D$ is true iff $d = 0$ and abc has odd parity (in which case $abcd$ has odd parity), or $d = 1$ and abc has even parity (in which case $abcd$ has odd parity).
   $(A \leftrightarrow B \leftrightarrow C) \leftrightarrow D$ is true iff $d = 1$ and abc has odd parity (in which case $abcd$ has even parity), or $d = 0$ and abc has even parity (in which case $abcd$ has even parity).
   Can you express your answer in terms of the parity of the string $abcd$?
   So $A \oplus B \oplus C \oplus D$ is true iff $abcd$ has odd parity
   $A \leftrightarrow B \leftrightarrow C \leftrightarrow D$ is true iff $abcd$ has even parity.

   (b) What about the case with two boolean variables?

   $$A \oplus B \quad \text{and} \quad A \leftrightarrow B$$

   When are these formulae true?
   $A \oplus B$ is true iff ab has odd parity;
   $A \leftrightarrow B$ is true iff ab has even parity.

   (c) What can you say about the case with five boolean variables?

   $$A \oplus B \oplus C \oplus D \oplus E \quad \text{and} \quad A \leftrightarrow B \leftrightarrow C \leftrightarrow D \leftrightarrow E$$

   When are these formulae true?
   For $\oplus$ this is just as before $A \oplus B \oplus C \oplus D \oplus E$ is true iff $abcde$ has odd parity.

[1] We say binary number has odd/even **parity** if it has an odd/even number of 1s.

$(A \leftrightarrow B \leftrightarrow C \leftrightarrow D) \leftrightarrow E$ is true iff $abcd$ has even parity and $e = 1$ (in which case $abcde$ has odd parity)
or $abcd$ has odd parity and $e = 0$ (in which case $abcde$ has odd parity)
Can you express your answer in terms of the parity of the string $abcde$?
Each is true iff $abcde$ has odd parity.

(d) The general case is to consider expressions with n variables

$$A_0 \oplus A_1 \oplus A_2 \oplus \cdots \oplus A_{n-3} \oplus A_{n-2} \oplus A_{n-1}$$

and

$$A_0 \leftrightarrow A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_{n-3} \leftrightarrow A_{n-2} \leftrightarrow A_{n-1}$$

Can you say when these formulae are true (the answer will depend on n)?

When n is odd, the two expressions are equivalent; each expression is true iff the corresponding binary number has odd parity.
When n is even, the expression with $\oplus$ is true iff the corresponding binary number has odd parity, while the expression with $\leftrightarrow$ is true iff the binary number has even parity.

(e) Does your general answer work when $n = 1$?
Yes

(f) How should we define the formulae for the case where $n = 0$?
If we define

$$\oplus(A_0, A_1, A_2, \ldots, A_{n-2}, A_{n-1}) \equiv A_0 \oplus A_1 \oplus A_2 \oplus \cdots \oplus A_{n-2} \oplus A_{n-1}$$
$$\leftrightarrow (A_0, A_1, A_2, \ldots, A_{n-2}, A_{n-1}) \equiv A_0 \leftrightarrow A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_{n-2} \leftrightarrow A_{n-1}$$

Then the required definitions are

$$\oplus() = \bot \qquad\qquad \leftrightarrow () = \top$$

since

$$\bot \oplus A = A \qquad\qquad \top \leftrightarrow A = A$$

Question (2f) is analogous to the question of how we define the factorial of 0. One reason to define $!0 = 1$, is so that the equation $!(n+1) = (n+1)!n$ holds when $n = 0$. Your general rule should be derived from equations that relate the case for formulae with $n + 1$ variables to the case for formulae with n variables. Once you have this relation, you can work backwards to define the appropriate formulae for the case $n = 0$.

3. Since $\leftrightarrow$ and $\oplus$ are commutative and associative, the boolean functions corresponding to the expressions studied in question (2) are **permutation invariant**.

   This means, for example, that the function f defined by

   $$f(a,b,c,d,e) = a \oplus b \oplus c \oplus d \oplus e$$

   returns the same value if we permute its arguments

   $$a \oplus b \oplus c \oplus d \oplus e = b \oplus a \oplus c \oplus d \oplus e = b \oplus c \oplus a \oplus e \oplus d \ldots$$
   $$f(a,b,c,d,e) = f(b,a,c,d,e) = f(b,c,a,e,d) \ldots \text{and so on}$$

   (a) Which of the three-digit binary numbers can be obtained by permuting the digits of 101?

   i. 000
   ii. 001
   iii. 010
   iv. 011•
   v. 100
   vi. 101•
   vii. 110•
   viii. 111

   Those with one 0 and two 1s.
   Note that the identity function is a permutation!

   (b) How many binary numbers can be obtained by permuting the digits of 10101? $10 = 5 \times 4/2 = 5 \times 4 \times 3/6$ – choose which are 0s or which are 1s.

   (c) What property do two binary numbers *abcde* and *vwxyz* have in common if they are are related by a permutation?
   They contain the same number of 1s (equivalently, the same number of 0s).

   (d) How many boolean functions of two boolean variables are there? How many of these are permutation invariant?
   There are $16 = 2^{2^2}$ boolean functions of two variables. The permutation invariant ones must must return true or false depending only on the number of 1s (zero, one or two) in the 2-digit argument, so there are $8 = 2^3$ permutation invariant functions.

   (e) How many boolean functions of two n boolean variables are there? How many of these are permutation invariant?
   There are 2^{2^n} boolean functions of n variables. The permutation invariant ones must return true or false depending only on the number of 1s (any number from 0 to n), so there are 2^{n+1} permutation invariant functions.

Summary of logical connectives

Symbol	Meaning	Example	Alternative symbols
$\neg$	not	$\neg A$	$\sim A$
$\wedge$	and	$A \wedge B$	$A\&B$
$\vee$	or	$A \vee B$	
$\rightarrow$	implies	$A \rightarrow B$	
$\leftrightarrow$	iff	$A \leftrightarrow B$	
$\oplus$	xor	$A \oplus B$	

This tutorial exercise sheet was written by Dave Cochran and Michael Fourman, and includes material from an earlier version by Mark McConville, revised by Paolo Besana, Thomas French, Areti Manataki, and Michael Fourman. Send comments to `michael.fourman@ed.ac.uk`