Informatics 1 - Computation & Logic: Tutorial 4

Satisfiability and Resolution

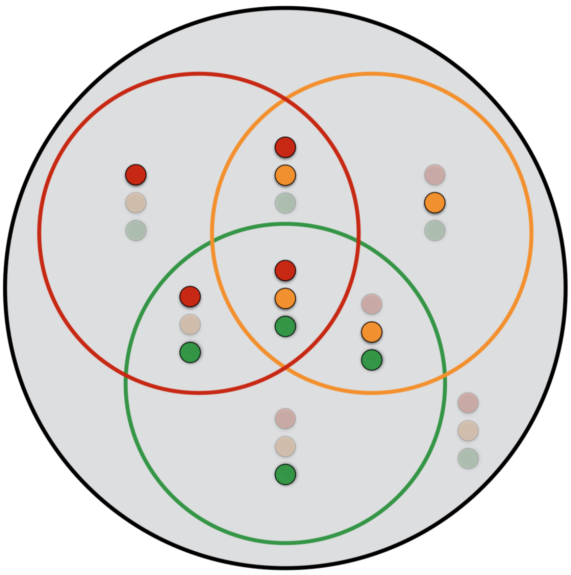
Week 6: 24-28 October 2016

Please attempt the entire worksheet in advance of the tutorial, and bring with you all work, including (if a computer is involved) printouts of code and test results. Tutorials cannot function properly unless you do the work in advance.

You may work with others, but you must understand the work; you can't phone a friend during the exam.

Assessment is formative, meaning that marks from coursework do not contribute to the final mark. But coursework is not optional. If you do not do the coursework you are unlikely to pass the exams.

Attendance at tutorials is **obligatory**; please let your tutor know if you cannot attend.



Consider the following clausal form:

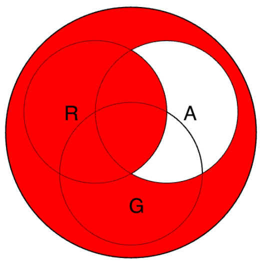
$$\{\,\{R, \neg A\}\,, \{G, A\}\,\}$$

Resolving on A, we get:

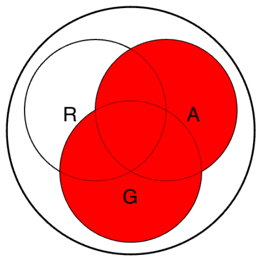
$$\frac{\{R, \neg A\} \qquad \{G, A\}}{\{R, G\}}\ (A)$$

It's worth remembering the reasoning behind resolution; If $(R \vee \neg A) \wedge (G \vee A)$ is true, if A is true, $\neg A$ must be false, and so for $R \vee \neg A$ to be true, R must be true; otherwise, A is false, and so for $G \vee A$ to be true, G must be true; therefore, $G \vee R$ must be true.

Recall the discussion in last week's tutorial sheet regarding clauses as *constraints*. We can visualise this resolution using Venn diagrams. Remember, here we are interested in the regions *excluded* by the clauses—the regions in white. Thus, the original clauses exclude:

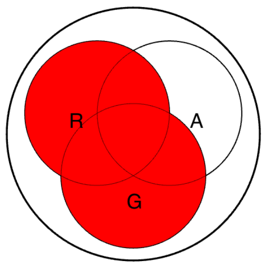


$\{R, \neg A\}$



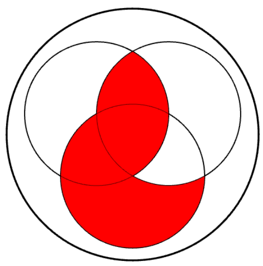
$\{G, A\}$

Our resolvent excludes (white regions):



$\{R, G\}$

Compare this to the valuations excluded by the whole clausal form (the union of the set of valuations excluded by the individual clauses):



$\{\,\{R, \neg A\}\,, \{G, A\}\,\}$

We see that the valuations excluded by the resolvent are a subset of the valuations excluded by the conjunction of the original clauses (resolvends); that is to say, the resolvent only excludes valuations that were already excluded by the resolvents; or again—*the resolvends entail the resolvent.*

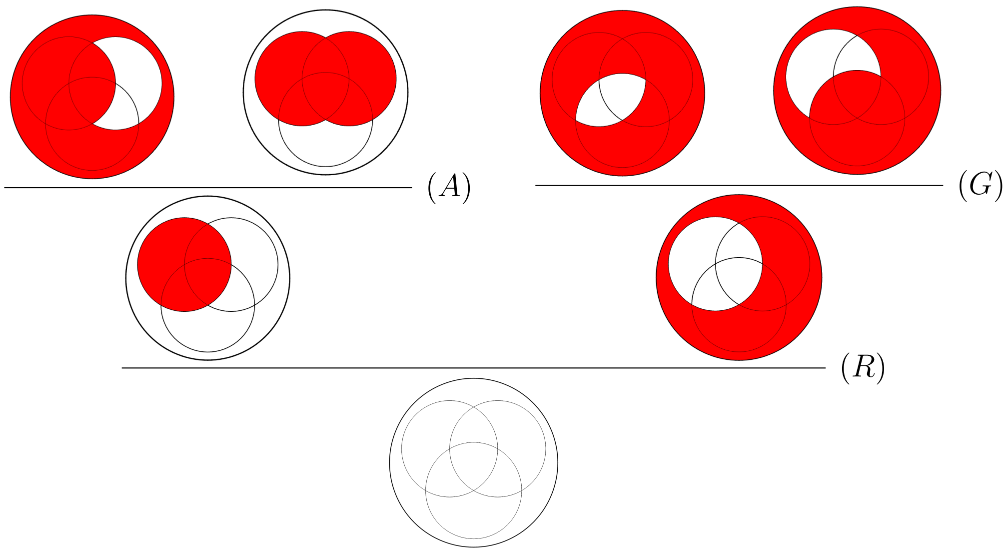
$$R \vee \neg A, G \vee A \vdash R \vee G$$

Now, let us consider a case that can be resolved to the empty set:

$$\{\, \{R, \neg A\}\, , \{R, A\}\, , \{\neg R, \neg G\}\, , \{\neg R, G\}\, \}$$

$$\dfrac{\dfrac{\{R, \neg A\} \qquad \{R, A\}}{\{R\}}\,(A) \qquad \dfrac{\{\neg R, \neg G\} \qquad \{\neg R, G\}}{\{\neg R\}}\,(G)}{\{\}}\,(R)$$

Using Venn diagrams:



Here, the resolution process shows us that *all* valuations are excluded by the conjunction of the clauses; thus we can see that the clausal form has *no* satisfying valuations; it is contradictory.[1]

[1] You may have noticed that here, the sets of valuations excluded by the resolvents are *identical to* the unions of the sets excluded by their resolvends; this happens when the resolvends differ *only* in the literal to be resolved upon.

1. Use resolution to show whether the following clausal forms are satisfiable or not. Use Venn diagrams to check your answers, including intermediate resolvents.

   (a) $\{\{A, \neg G\}, \{G, , \neg R\}, \{R\}, \{\neg A\}\}$

(b) $\big\{\{R, A\}, \{\neg R, \neg A, \neg G\}\big\}, \{G\}\big\}$

A Karnaugh Map (or *K-map*), like a Venn diagram, is a visual representation of a boolean expression. For domains consisting of four boolean letters, the map may be presented as a four-by four grid—but note that the two-dimensional plane wraps around on itself, from its bottom edge to its top, and from left to right - like the screen in Pac-Man or Asteroids. This geometry may be represented on the surface of a torus.

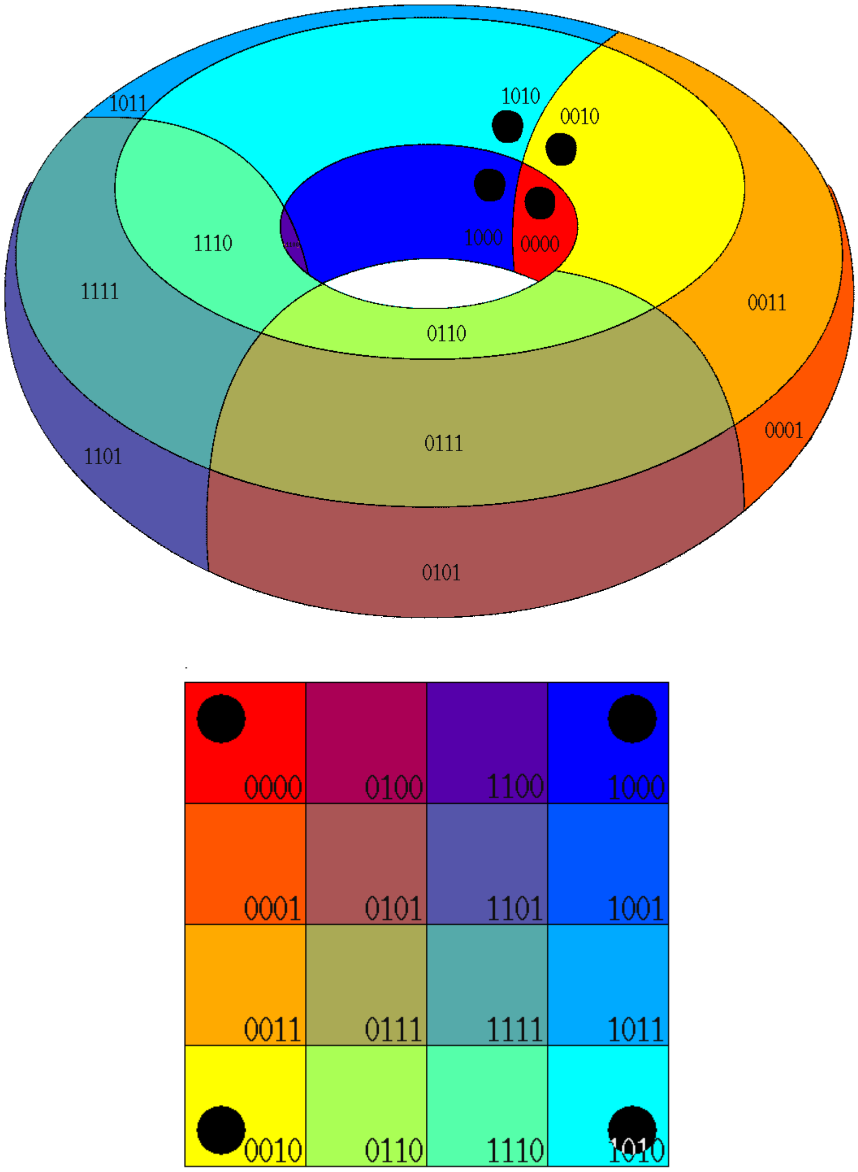
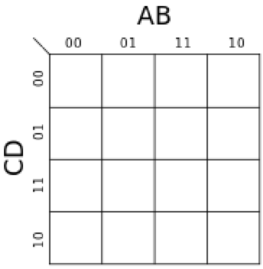


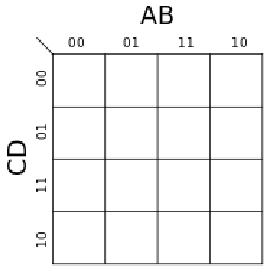
Figure 1: A Karnaugh map, wrapped around a torus and flattened out as a 4×4 grid. Note that the four dotted corners on the grid are adjacent to each other, as shown on the torus. The four-bit numbers show the state $ABCD$ represented by each square .

A K-map is often filled in with a 1 or 0 in each square, to indicate the truth or falsity, in each state, of a Boolean function of four variables. You can use ones and zeros, or colours or your choice, to indicate truth and falsity.

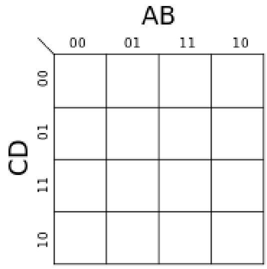
2. First fill in the Karnaugh maps below to show the area corresponding to (the truth of) each of the atomic propositions, and the negations of two of them.



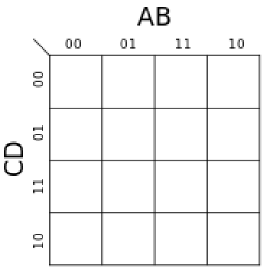
A



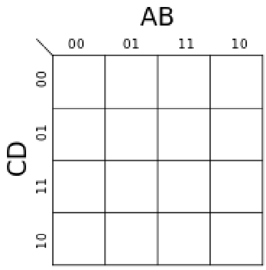
$\neg A$



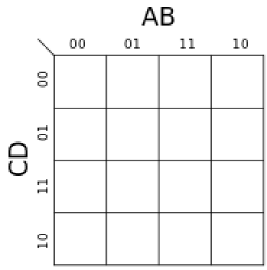
B



C



D

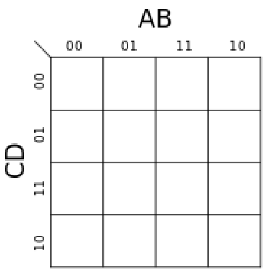


$\neg D$

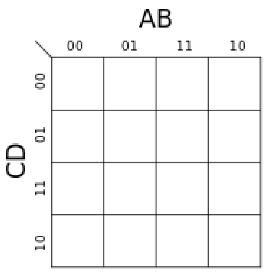
3. Consider the clausal form:

$$\{\,\{D, B\}\,, \{A, \neg B\}\,, \{\neg A, \neg B\}\,, \{\neg C, \neg D\}\,, \{C, \neg D\}\,\}$$

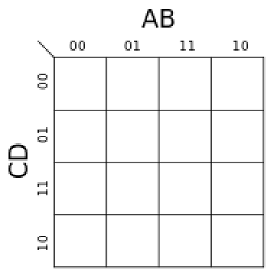
   (a) Show the region corresponding each of the clauses in its own K-map:



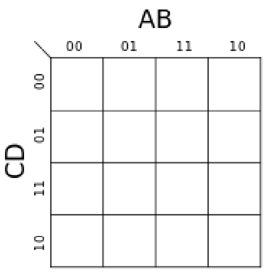
$\{D, B\}$



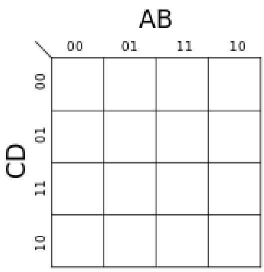
$\{A, \neg B\}$



$\{\neg A, \neg B\}$

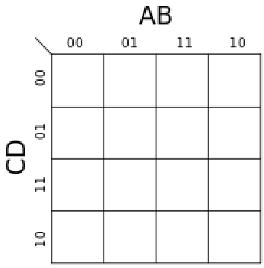


$\{\neg C, \neg D\}$

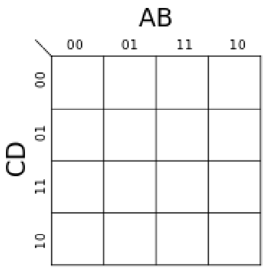


$\{C, \neg D\}$

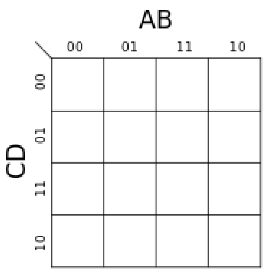
   (b) Now, take all clauses containing A or $\neg A$, and resolve on A every possible pairing. Cross out the resolvends, and show the resolvent(s) in on the provided K-map. The remaining clauses from the original clausal form plus the new resolvents are your *resolution pool*.



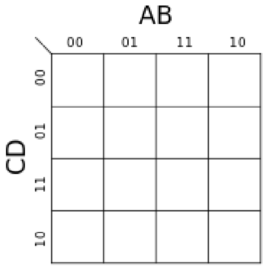
(c) If the previous step did not result in the empty clause being found, repeat the procedure, this time resolving on B, again using every pair of clauses in your resolution pool with complementary B-literals. Again, cross out the resolvends. The remaining clauses, plus the new resolvents are your updated resolution pool.



(d) If the previous step did not result in the empty clause being found, repeat resolving on C.



(e) If the previous step did not result in the empty clause being found, repeat resolving on D.



(f) Was the empty clause found?

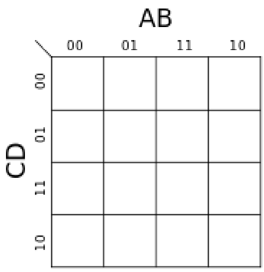
(g) Is the clausal form satisfiable?

The previous example was somewhat trivial, because at each stage you in fact only had to resolve one pair of resolvends. This will not always be the case. Thus, if you have three clauses containing C and three containing $\neg C$, you will have nine resolutions to perform on C. In the worst case scenario, this can be computationally intensive for large clausal forms; however, it has the advantage of guaranteeing that once each atom has been resolved upon, if the clauses are not satisfiable, the empty clause *will* have been found.

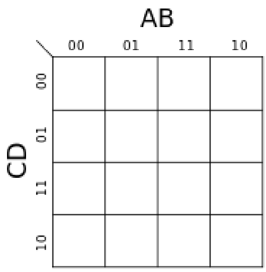
4. Consider the following clausal form:

$$\{\, \{A\}, \{B, \neg D\}, \{\neg A, \neg B, C, \neg D\}, \{\neg A, D\} \,\}$$

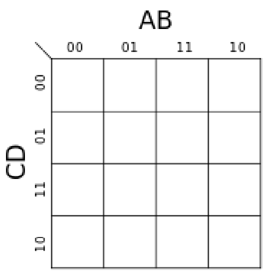
(a) Again, show the region corresponding to each clause in its own K-map:



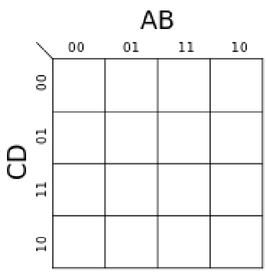
$\{A\}$



$\{B, \neg D\}$



$\{\neg A, \neg B, C, \neg D\}$



$\{\neg A, D\}$

(b) Following the same procedure as before, resolve on:

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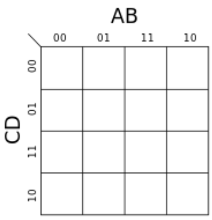
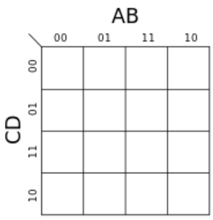
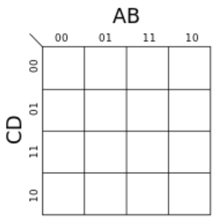
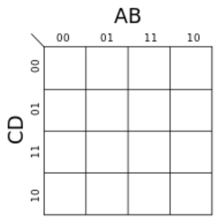
			

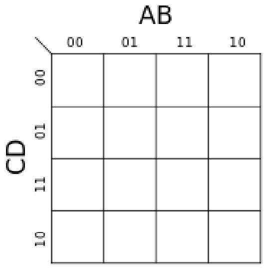
(c) Was the empty clause found?

(d) Is the clausal form satisfiable?

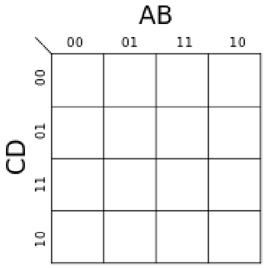
5. Our next example includes 6 clauses:

$$\{\,\{A, B\}\,, \{A, \neg B, \neg C\}\,, \{\neg A, D\}\,, \{\neg B, C, D\}\,, \{\neg B, \neg D\}\,, \{\neg A, B, \neg D\}\,\}$$

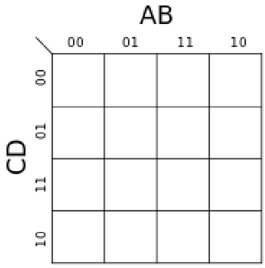
(a) Show each clause in its own K-map:



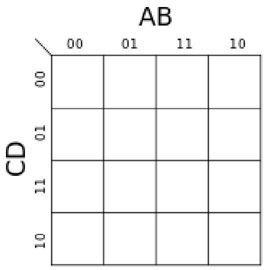
$\{A, B\}$



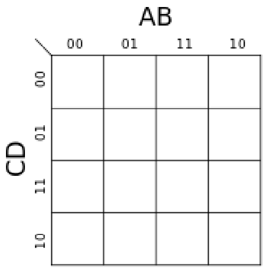
$\{A, \neg B, \neg C\}$



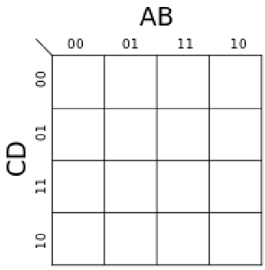
$\{\neg A, D\}$



$\{\neg B, C, D\}$

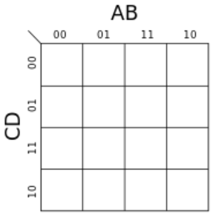
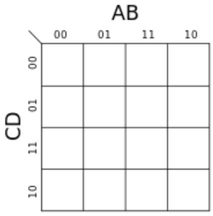
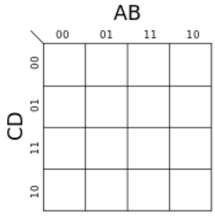
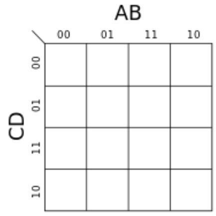


$\{\neg B, \neg D\}$



$\{\neg A, B, \neg D\}$

(b) Following the same procedure as before, resolve on:

A Resolvents:	B Resolvents:	C Resolvents:	C Resolvents:
			

(c) Was the empty clause found?

(d) Is the clausal form satisfiable?

6. Of course, logical claims can consist of arbitrarily many atoms, and our ability to usefully represent them visually is eventually exhausted. However, resolution can be applied to sets of clauses with any number of atoms. This question concerns the resolution of the claim that:

$$P \to (Q \lor R), Q \to \neg S, S \lor R, R \to Q, (Q \land R) \to T \vdash P \to T$$

(a) Express each of the assumptions, and the negation of the conclusion, in clausal form.

i. $P \to (Q \lor R)$

ii. $Q \to \neg S$

iii. $S \lor R$

iv. $R \to Q$

v. $(Q \land R) \to T$

vi. $\neg(P \to T)$

(b) Use resolution to determine whether the negation of the conclusion is consistent with the conjunction of the assumptions.

(c) Is the original claim correct?

7. This question concerns the 256 possible truth valuations of the following eight propositional letters A, B, C, D, E, F, G, H. For each of the following expressions, say how many of the 256 valuations satisfy the expression, and briefly explain your reasoning. For example, the expression D is satisfied by half of the valuations, that is 128 of the 256, since for each valuation that makes D true there is a matching valuation that make D false.

   (a) $A \wedge B$

   (b) $(A \vee B) \wedge C$

   (c) $(A \to B) \to C$

   (d) $(A \to B) \wedge (B \to A) \wedge (C \to D) \wedge (D \to E) \wedge (E \to F) \wedge (F \to G) \wedge (G \to H)$

(e)

$$(A \to B) \wedge (B \to A) \wedge (C \to D) \wedge (D \to C) \\ \wedge (E \to F) \wedge (F \to G) \wedge (G \to H)$$

(f)

$$(H \to A) \wedge (A \to B \wedge C) \wedge (B \vee C \to D) \wedge (A \to E) \wedge (E \to F) \wedge (F \to G) \wedge (G \to H)$$

This tutorial exercise sheet was written by Dave Cochran and Michael Fourman, with additional contributions from an earlier tutorials produced by Paolo Besana, Thomas French, and Areti Manataki. Send comments to `Michael.Fourman@ed.ac.uk`