

Informatics 1 - Computation & Logic: Tutorial 8 Solutions

Computation: Subset Procedure

Week 10: 18-22 November 2013

The Subset Procedure

Recall the following definition:

A finite state machine is an ordered 5-tuple $\langle Q, \Sigma, s_0, F, \delta \rangle$, where:

- Q is a finite set of states;
- Σ is an alphabet of input symbols;
- $s_0 \in Q$ is the initial state;
- $F \subseteq Q$ is the set of accept states; and
- δ is a set of transitions, essentially a subset of $Q \times \Sigma \times Q$.

A finite state machine $\langle Q, \Sigma, s_0, F, \delta \rangle$ is deterministic if and only if for every $q \in Q$ and every $\sigma \in \Sigma$, there is just one transition $\langle q, \sigma, q' \rangle \in \delta$.

Every non-deterministic FSM can be converted into a deterministic FSM that accepts exactly the same set of strings, by means of an algorithm known as the *subset procedure*.

The formal definition states that the result of applying the subset procedure to FSM $\langle Q, \Sigma, s_0, F, \delta \rangle$ is the deterministic FSM $\langle Q', \Sigma, \{s_0\}, F', \delta' \rangle$, where:

- Q' is a set of subsets of Q i.e. superstates
- F' is the set of all and only superstates in Q' which contain at least one state in F
- δ' is the set of derived transitions — given the original FSM $\langle Q, \Sigma, s_0, F, \delta \rangle$, in the new, derived machine, there is a transition from superstate A by means of symbol $\sigma \in \Sigma$ to the following superstate:

$$\{q \mid \text{for some } q' \in A, \langle q', \sigma, q \rangle \in \delta\}$$

i.e. the set of all states in the original machine that can be reached from *some* state in A by means of a transition labelled σ .

Take for example the following FSM, which accepts all and only the strings over $\{b, c\}$ that begin with b and end with c :

$$\mathcal{M}_1 = \langle \{0, 1, 2\}, \{b, c\}, 0, \{2\}, \{\langle 0, b, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, c, 1 \rangle, \langle 1, c, 2 \rangle\} \rangle$$

It may help to draw out this FSM as a diagram. This machine is non-deterministic because there are two distinct transitions from state 1 for the symbol c .

Let us now consider how to convert \mathcal{M}_1 into a deterministic FSM, \mathcal{M}_2 , that accepts exactly the same set of strings.

First of all, note that the set of states in \mathcal{M}_1 is $\{0, 1, 2\}$. The states in the deterministic machine \mathcal{M}_2 will be *subsets* of those states, which we call *superstates*.

Step 1: Identify initial superstate

The first step in converting \mathcal{M}_1 into a deterministic machine is to identify the initial superstate of the new machine. To do this we just take the initial state of \mathcal{M}_1 , i.e. 0, and convert it into a superstate containing only this state — $\{0\}$. Thus, the new machine, \mathcal{M}_2 , starts off as follows:

$$\mathcal{M}_2 = \langle \{\{0\}\}, \{b, c\}, \{0\}, \emptyset, \emptyset \rangle$$

Again, it might be useful to draw this partial machine. Note that as of yet the machine \mathcal{M}_2 does not contain any accept states or transitions. It has just the single state, $\{0\}$, that is also the initial state.

Step 2: Incrementally add transitions

The next part of the subset procedure involves incrementally building up the partial FSM until every superstate has exactly one transition for each symbol in the alphabet $\{b, c\}$. First, we identify those transitions that are missing — the partial machine has one state $\{0\}$ and this state lacks two transitions: one for symbol b and one for symbol c . Thus we must find:

- a transition from superstate $\{0\}$ for symbol b , and,
- a transition from superstate $\{0\}$ for symbol c

Given some non-deterministic FSM $\langle Q, \Sigma, s_0, F, \delta \rangle$, a superstate $A \in \wp(Q)$ and a symbol $\sigma \in \Sigma$, the superstate in the new machine that is reached from superstate A by means of symbol σ is defined as follows:

$$\{q \mid \text{for some } q' \in A, \langle q', \sigma, q \rangle \in \delta\}$$

Thus:

- the transition from superstate $\{0\}$ for symbol b is:

$$\{q \mid \text{for some } q' \in \{0\}, \langle q', b, q \rangle \in \{\langle 0, b, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, c, 1 \rangle, \langle 1, c, 2 \rangle\}\} = \{1\}$$

- the transition from superstate $\{0\}$ for symbol c is:

$$\{q \mid \text{for some } q' \in \{0\}, \langle q', c, q \rangle \in \{\langle 0, b, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, c, 1 \rangle, \langle 1, c, 2 \rangle\}\} = \emptyset$$

Thus, we have two new transitions to add to our partial machine: $\langle \{0\}, b, \{1\} \rangle$ and $\langle \{0\}, c, \emptyset \rangle$. Adding the new transitions and the new superstates $\{1\}$ and \emptyset to \mathcal{M}_2 , the partial machine is now:

$$\mathcal{M}_2 = \langle \{\{0\}, \{1\}, \emptyset\}, \{b, c\}, \{0\}, \emptyset, \{\langle \{0\}, b, \{1\} \rangle, \langle \{0\}, c, \emptyset \rangle\} \rangle$$

This partial machine still has transitions missing, so we continue the procedure:

- the transition from superstate $\{1\}$ for symbol b is:

$$\{q \mid \text{for some } q' \in \{1\}, \langle q', b, q \rangle \in \{\langle 0, b, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, c, 1 \rangle, \langle 1, c, 2 \rangle\}\} = \{1\}$$

- the transition from superstate $\{1\}$ for symbol c is:

$$\{q \mid \text{for some } q' \in \{1\}, \langle q', c, q \rangle \in \{\langle 0, b, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, c, 1 \rangle, \langle 1, c, 2 \rangle\}\} = \{1, 2\}$$

- the transition from superstate \emptyset for symbol b is:

$$\{q \mid \text{for some } q' \in \emptyset, \langle q', b, q \rangle \in \{\langle 0, b, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, c, 1 \rangle, \langle 1, c, 2 \rangle\}\} = \emptyset$$

- the transition from superstate \emptyset for symbol c is:

$$\{q \mid \text{for some } q' \in \emptyset, \langle q', c, q \rangle \in \{\langle 0, b, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, c, 1 \rangle, \langle 1, c, 2 \rangle\}\} = \emptyset$$

Thus the superstate $\{1, 2\}$ is added to machine \mathcal{M}_2 , along with four new transitions: $\langle \{1\}, b, \{1\} \rangle$, $\langle \{1\}, c, \{1, 2\} \rangle$, $\langle \emptyset, b, \emptyset \rangle$, and $\langle \emptyset, c, \emptyset \rangle$. The partial machine is now (again, it may be useful to draw out the partial machine):

$$\mathcal{M}_2 = \langle \{\{0\}, \{1\}, \emptyset, \{1, 2\}\}, \{b, c\}, \{0\}, \emptyset, \delta \rangle$$

where $\delta = \{\langle \{0\}, b, \{1\} \rangle, \langle \{0\}, c, \emptyset \rangle, \langle \{1\}, b, \{1\} \rangle, \langle \{1\}, c, \{1, 2\} \rangle, \langle \emptyset, b, \emptyset \rangle, \langle \emptyset, c, \emptyset \rangle\}$

There is *still* one superstate $\{1, 2\}$ which lacks the appropriate transitions, so we continue:

- the transition from superstate $\{1, 2\}$ for symbol b is:

$$\{q \mid \text{for some } q' \in \{1, 2\}, \langle q', b, q \rangle \in \{\langle 0, b, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, c, 1 \rangle, \langle 1, c, 2 \rangle\}\} = \{1\}$$

- the transition from superstate $\{1, 2\}$ for symbol c is:

$$\{q \mid \text{for some } q' \in \{1, 2\}, \langle q', c, q \rangle \in \{\langle 0, b, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, c, 1 \rangle, \langle 1, c, 2 \rangle\}\} = \{1, 2\}$$

So we add a further two transitions to \mathcal{M}_2 : $\langle\{1, 2\}, b, \{1\}\rangle$ and $\langle\{1, 2\}, c, \{1, 2\}\rangle$. The machine is now:

$$\mathcal{M}_2 = \langle\{\{0\}, \{1\}, \emptyset, \{1, 2\}\}, \{b, c\}, \{0\}, \emptyset, \delta_2\rangle$$

where

$$\delta_2 = \{\langle\{0\}, b, \{1\}\rangle, \langle\{0\}, c, \emptyset\rangle, \langle\{1\}, b, \{1\}\rangle, \langle\{1\}, c, \{1, 2\}\rangle, \langle\emptyset, b, \emptyset\rangle, \langle\emptyset, c, \emptyset\rangle, \langle\{1, 2\}, b, \{1\}\rangle, \langle\{1, 2\}, c, \{1, 2\}\rangle\}$$

The transition set of the new machine is now complete, since every superstate has exactly one transition for each symbol in the alphabet $\{b, c\}$.

Step 3: Identify accept superstates

The only thing that remains to do is to identify the set of accept states in the new, deterministic FSM. Basically, *all* superstates in the new FSM that contains at least *one* of the accept states in the original non-deterministic FSM are accept states of the new, derived machine. The set of accept states in \mathcal{M}_1 is $\{2\}$, so the only accept state in \mathcal{M}_2 is the superstate $\{1, 2\}$.

Step 4: Full definition of deterministic FSM

In conclusion, the result of applying the subset procedure to the non-deterministic FSM \mathcal{M}_1 is the following *deterministic* machine \mathcal{M}_2 :

$$\mathcal{M}_2 = \langle\{\{0\}, \{1\}, \emptyset, \{1, 2\}\}, \{b, c\}, \{0\}, \{\{1, 2\}\}, \delta_2\rangle$$

where

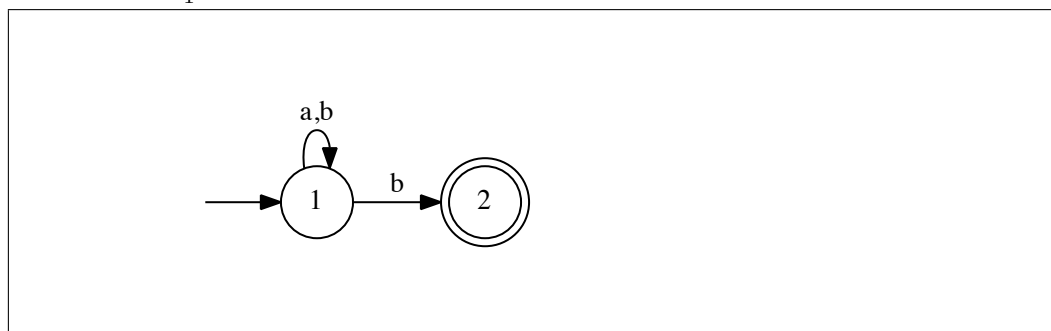
$$\delta_2 = \{\langle\{0\}, b, \{1\}\rangle, \langle\{0\}, c, \emptyset\rangle, \langle\{1\}, b, \{1\}\rangle, \langle\{1\}, c, \{1, 2\}\rangle, \langle\emptyset, b, \emptyset\rangle, \langle\emptyset, c, \emptyset\rangle, \langle\{1, 2\}, b, \{1\}\rangle, \langle\{1, 2\}, c, \{1, 2\}\rangle\}$$

- Using the subset procedure, convert the following non-deterministic FSM:

$$\mathcal{M}_1 = \langle\{1, 2\}, \{a, b\}, 1, \{2\}, \{\langle 1, a, 1\rangle, \langle 1, b, 1\rangle, \langle 1, b, 2\rangle\}\rangle$$

to a deterministic FSM, \mathcal{M}_2 , that accepts the same language.

- Draw the \mathcal{M}_1 machine:



(b) Give the complete set of states, Q_2 , of \mathcal{M}_2 :

$\{\{1\}, \{1, 2\}\}$

(c) What is the start state, s_0 , of \mathcal{M}_2 ?

$\{1\}$

(d) Give the set of transitions, δ_2 , of \mathcal{M}_2 :

$\{\langle\{1\}, a, \{1\}\rangle, \langle\{1\}, b, \{1, 2\}\rangle, \langle\{1, 2\}, a, \{1\}\rangle, \langle\{1, 2\}, b, \{1, 2\}\rangle\}$

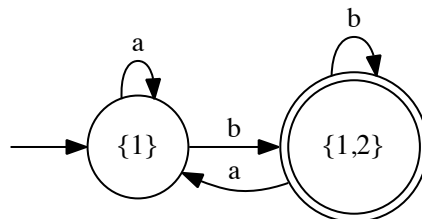
(e) What is the set of accepting states, F_2 , of \mathcal{M}_2 ?

$\{\{1, 2\}\}$

(f) Now give the full definition of \mathcal{M}_2 :

$\mathcal{M}_2 = \langle\{\{1\}, \{1, 2\}\}, \{a, b\}, \{1\}, \{\{1, 2\}\}, \delta_2\rangle$
 $\delta_2 = \{\langle\{1\}, a, \{1\}\rangle, \langle\{1\}, b, \{1, 2\}\rangle, \langle\{1, 2\}, a, \{1\}\rangle, \langle\{1, 2\}, b, \{1, 2\}\rangle\}$

(g) Finally, draw the \mathcal{M}_2 machine:

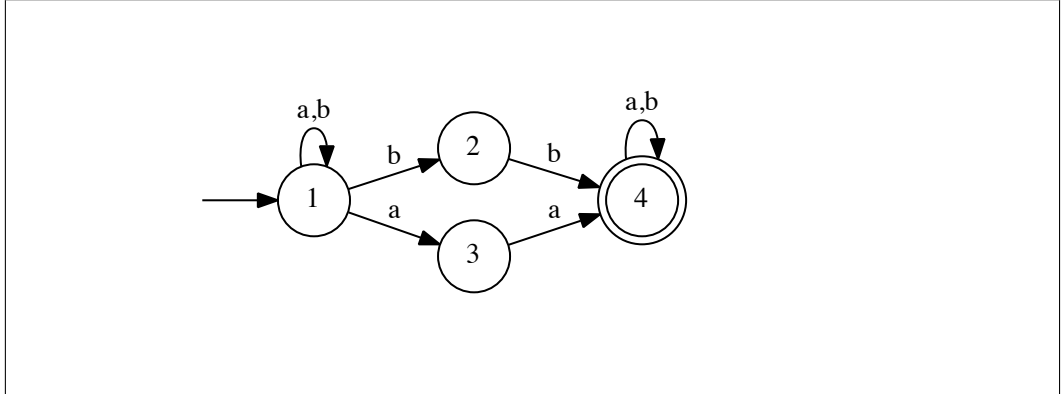


2. Convert the following non-deterministic FSM:

$$\mathcal{M}_3 = \langle \{1, 2, 3, 4\}, \{a, b\}, 1, \{4\}, \delta_3 \rangle$$

$\delta_3 = \{ \langle 1, a, 1 \rangle, \langle 1, b, 1 \rangle, \langle 1, b, 2 \rangle, \langle 1, a, 3 \rangle, \langle 2, b, 4 \rangle, \langle 3, a, 4 \rangle, \langle 4, a, 4 \rangle, \langle 4, b, 4 \rangle \}$ to a deterministic FSM, \mathcal{M}_4 , that accepts the same language.

(a) Draw the \mathcal{M}_3 machine:



(b) What is the start state, s_0 , of \mathcal{M}_4 ?

{1}

(c) Give the set of transitions, δ_4 , of \mathcal{M}_4 :

$\{ \langle \{1\}, a, \{1, 3\} \rangle, \langle \{1\}, b, \{1, 2\} \rangle, \langle \{1, 3\}, a, \{1, 3, 4\} \rangle, \langle \{1, 3\}, b, \{1, 2\} \rangle, \langle \{1, 2\}, a, \{1, 3\} \rangle, \langle \{1, 2\}, b, \{1, 2, 4\} \rangle, \langle \{1, 3, 4\}, a, \{1, 3, 4\} \rangle, \langle \{1, 3, 4\}, b, \{1, 2, 4\} \rangle, \langle \{1, 2, 4\}, a, \{1, 3, 4\} \rangle, \langle \{1, 2, 4\}, b, \{1, 2, 4\} \rangle \}$

(d) Give the set of states, Q_4 , of \mathcal{M}_4 :

$\{ \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 4\}, \{1, 3, 4\} \}$

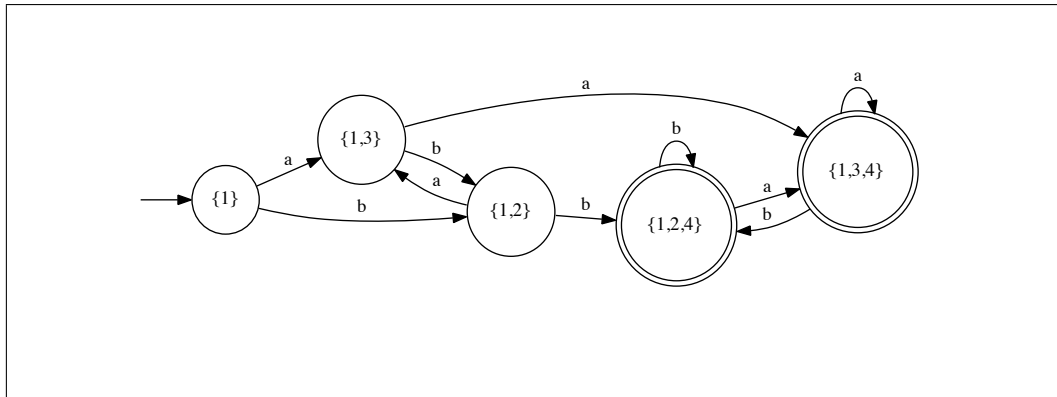
(e) What is the set of accepting states, F_4 , of \mathcal{M}_4 ?

$\{ \{1, 2, 4\}, \{1, 3, 4\} \}$

(f) Now give the full definition of \mathcal{M}_4 :

$\mathcal{M}_4 = \langle \{ \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 4\}, \{1, 3, 4\} \}, \{a, b\}, \{1\}, \{ \{1, 2, 4\}, \{1, 3, 4\} \}, \delta_4 \rangle$
 $\delta_4 = \{ \langle \{1\}, a, \{1, 3\} \rangle, \langle \{1\}, b, \{1, 2\} \rangle, \langle \{1, 3\}, a, \{1, 3, 4\} \rangle, \langle \{1, 3\}, b, \{1, 2\} \rangle, \langle \{1, 2\}, a, \{1, 3\} \rangle, \langle \{1, 2\}, b, \{1, 2, 4\} \rangle, \langle \{1, 3, 4\}, a, \{1, 3, 4\} \rangle, \langle \{1, 3, 4\}, b, \{1, 2, 4\} \rangle, \langle \{1, 2, 4\}, a, \{1, 3, 4\} \rangle, \langle \{1, 2, 4\}, b, \{1, 2, 4\} \rangle \}$

(g) Finally, draw the \mathcal{M}_4 machine:



This tutorial exercise sheet was written by Mark McConville, revised by Paolo Besana, and extended by Thomas French and Areti Manataki. Send comments to A.Manataki@ed.ac.uk