Informatics 1 - Computation & Logic: Tutorial 4

Propositional Logic: Resolution

Week 6: 21-25 October 2013

Please attempt the entire worksheet in advance of the tutorial, and bring with you all work, including (if a computer is involved) printouts of code and test results. Tutorials cannot function properly unless you do the work in advance.

You may work with others, but you must understand the work; you can't phone a friend during the exam.

Assessment is formative, meaning that marks from coursework do not contribute to the final mark. But coursework is not optional. If you do not do the coursework you are unlikely to pass the exams.

Attendance at tutorials is **obligatory**; please let your tutor know if you cannot attend.

Resolution

The following propositional logic argument is valid:

$$(P \wedge Q) \to R,\ P,\ Q \vdash R \tag{1}$$

In Tutorials 2 and 3, you learned two methods for proving that an argument is valid: (1) using truth tables; (2) using sequent proofs. In this assignment we are concerned with a third method: conversion to *clausal form* and *resolution*.

The resolution method proceeds in FOUR steps:

(1) **Convert the argument into a conjunction**

Recall from the lecture notes that argument $X_1, X_2, \ldots, X_n \vdash Y$ is valid if and only if the conjunction $X_1 \wedge X_2 \wedge \ldots \wedge X_n \wedge \neg Y$ is inconsistent (i.e. the negation of the conclusion is inconsistent with the premises). Thus we start by converting the argument in (1) into the following conjunction:

$$((P \wedge Q) \to R) \wedge P \wedge Q \wedge \neg R \tag{2}$$

If we can prove that this conjunction is inconsistent, then we have proved that the argument in (1) is valid.

(2) **Convert the conjunction into conjunctive normal form**

An expression is in conjunctive normal form (CNF) if it is a *conjunction of disjunctions of literals*, where a literal is either an atomic propositional symbol or a negated atomic propositional symbol.
To be more precise, an expression is in CNF if it is a conjunction of *one or more* disjunctions of *one or more* literals.
To be even more precise, an expression is in CNF if it is either: (a) a literal; (b) a disjunction of literals; or (c) a conjunction of literals and/or disjunctions of literals.
For example, the following expressions are all in CNF:

$$\begin{array}{l} \neg P \\ P \vee \neg Q \vee \neg R \\ P \ \wedge\ (\neg Q \vee \neg R) \\ P \ \wedge\ \neg Q \ \wedge\ \neg R \\ (P \vee \neg Q) \ \wedge\ (\neg R \vee P) \\ (P \vee Q \vee \neg R) \ \wedge\ (\neg Q \vee \neg R) \ \wedge\ P \ \wedge\ (\neg S \vee \neg P) \end{array}$$

However the following expressions are *not* in CNF:

$$\begin{array}{l} (P \vee \neg Q) \ \wedge\ (\neg R \rightarrow P) \\ (P \ \wedge\ \neg Q) \vee (\neg R \ \wedge P) \\ \neg P \ \wedge\ (\neg\neg Q \vee R) \\ (P \vee \neg Q) \ \wedge\ \neg(R \vee P) \end{array}$$

To convert an arbitrary expression of propositional logic into CNF, we apply the following equivalences:

- $X \leftrightarrow Y$ is equivalent to $(X \rightarrow Y) \ \wedge\ (Y \rightarrow X)$
- $X \rightarrow Y$ is equivalent to $\neg X \vee Y$
- $\neg(X \vee Y)$ is equivalent to $\neg X \ \wedge\ \neg Y$
- $\neg(X \ \wedge\ Y)$ is equivalent to $\neg X \vee \neg Y$
- $X \vee (Y \ \wedge\ Z)$ is equivalent to $(X \vee Y) \ \wedge\ (X \vee Z)$
- $X \wedge (Y \ \vee\ Z)$ is equivalent to $(X \wedge Y) \ \vee\ (X \wedge Z)$
- $\neg\neg X$ is equivalent to X

As well as making liberal use of the associativity conventions for conjunction and disjunction:

- $X \wedge (Y \wedge Z)$ is equivalent to $(X \wedge Y) \wedge Z$ and thus can be written $X \wedge Y \wedge Z$
- $X \vee (Y \vee Z)$ is equivalent to $(X \vee Y) \vee Z$ and thus can be written $X \vee Y \vee Z$

Thus we can convert the conjunction in (2) into CNF as follows:

$$\begin{array}{ll} & \underline{((P \wedge Q) \rightarrow R)} \wedge P \wedge Q \wedge \neg R \\ \Rightarrow & (\neg \underline{(P \wedge Q)} \vee R) \wedge P \wedge Q \wedge \neg R \\ \Rightarrow & (\neg P \vee \neg Q \vee R) \wedge P \wedge Q \wedge \neg R \end{array}$$

In other words, the conjunction in (2) is logically equivalent to the following CNF expression:

$$(\neg P \vee \neg Q \vee R) \wedge P \wedge Q \wedge \neg R \tag{3}$$

You can verify this using a truth table.

(3) Convert the CNF expression into clausal form

To turn a CNF expression into clausal form, simply turn each conjunct into a *set* of literals, and then convert the whole conjunction into a *set of sets* of literals.

The CNF expression in (3) can be converted into clausal form as follows:

$$\begin{array}{ll} & (\neg P \vee \neg Q \vee R) \wedge P \wedge Q \wedge \neg R \\ \Rightarrow & [\neg P, \neg Q, R] \wedge [P] \wedge [Q] \wedge [\neg R] \\ \Rightarrow & [[\neg P, \neg Q, R]\,,\ [P]\,,\ [Q]\,,\ [\neg R]] \end{array}$$

Thus the clausal form of the CNF expression in (3) is the following:

$$[[\neg P, \neg Q, R]\,,\ [P]\,,\ [Q]\,,\ [\neg R]] \tag{4}$$

(4) Apply the resolution rule to the expression in clausal form until no literals are left

A simple application of the resolution rule is as follows, where X, A and B are literals:

$$\frac{[[X, A]\,, [\neg X, B]]}{[[A, B]]}$$

In words, the complimentary literals, here X and $\neg X$, from the different clauses are removed, and the remaining literals from the two clauses, here A and B, are merged into a new clause.

More generally, the resolution rule is as follows, where $\mathcal{A}$ and $\mathcal{B}$ are sets of literals, and $\mathscr{C}$ is a set of clauses:

$$\frac{[[[X] \cup \mathcal{A}, [\neg X] \cup \mathcal{B}] \cup \mathscr{C}]}{[[\mathcal{A} \cup \mathcal{B}] \cup \mathscr{C}]}$$

If we apply the resolution rule to the clausal form in (4), one derivation is:

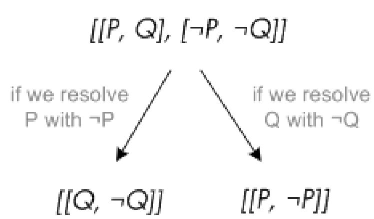
$$\begin{aligned} & [[\underline{\neg P}, \neg Q, R]\,,\ [\underline{P}]\,,\ [Q]\,,\ [\neg R]] \\ \Rightarrow\ & [[\underline{\neg Q}, R]\,,\ [\underline{Q}]\,,\ [\neg R]] \\ \Rightarrow\ & [[\underline{R}]\,,\ [\underline{\neg R}]] \\ \Rightarrow\ & [] \end{aligned}$$

Note that to show an argument is valid it is only required to give a *single derivation* of the empty clause; whereas to prove an argument is invalid it is necessary to exhaustively apply the resolution rule to *all combinations* of complimentary literals. This can be highly inefficient.

An example that demonstrates this follows. Given the expression:

$$[[P, Q]\,,\ [\neg P, \neg Q]] \tag{5}$$

we apply the resolution rule in the following way:



Here we exhaustively applied the resolution rule to all possible combinations of literals to prove that the argument that corresponds to (5) is invalid.

A more efficient algorithm for applying the resolution rule is the **Davis-Putnam algorithm**. In this, at each stage some variable is picked, and each clause containing positive occurrences of that variable is resolved with each clause containing negative occurrences of that variable.

If we apply the Davis-Putnam algorithm for resolution to the clausal form in (5), we get:

$$\begin{aligned} & [[\underline{P}, Q]\,,\ [\underline{\neg P}, \neg Q]] \\ \Rightarrow\ & [[Q, \neg Q]] \end{aligned}$$

This is considerably shorter than the previous solution. To make the most of the Davis-Putnam algorithm, you are encouraged to first pick variables that occur in few clauses.

1. Given the following arguments:

   (a) $C \vdash \neg\neg A \to B$

      i. Convert to conjunctions:

      ii. Convert to CNF:

   (b) $A \to B \vdash A \wedge B$

      i. Convert to conjunctions:

      ii. Convert to CNF:

   (c) $\neg(B \vee C) \vdash A \vee \neg B$

      i. Convert to conjunctions:

      ii. Convert to CNF:

(d) $\neg(\neg A \vee C), B \rightarrow (D \wedge C) \vdash A \wedge B$

i. Convert to conjunctions:

ii. Convert to CNF:

(e) $B \vee \neg E, C \leftrightarrow D \vdash A \rightarrow (B \rightarrow \neg C)$

i. Convert to conjunctions:

ii. Convert to CNF:

2. Use resolution to prove whether the following argument is valid:

$$\neg A \to \neg B, (\neg B \land A) \to D \vdash D$$

(a) Convert into conjunctions:

(b) Convert to CNF:

(c) Convert into clausal form:

(d) Apply the Davis-Putnam algorithm for resolution and state whether the original argument is valid:

The argument is VALID/INVALID

3. Use resolution to prove whether the following argument is valid:

$$\neg F \to \neg P,\ (\neg P \land Q) \to R \vdash \neg F \to (R \land \neg Q)$$

(a) Convert into conjunctions:

(b) Convert to CNF:

(c) Convert into clausal form:

(d) Apply the Davis-Putnam algorithm for resolution and state whether the original argument is valid:

The argument is VALID/INVALID

4. Use resolution to prove whether the following argument is valid:

$$A \to \neg C, (\neg B \lor D) \to A \vdash (D \land \neg B) \to (A \land \neg C)$$

(a) Convert to conjunctions:

(b) Convert to CNF:

(c) Convert in clausal form:

(d) Apply the Davis-Putnam algorithm for resolution and state whether the original argument is valid:

The argument is VALID/INVALID

This tutorial exercise sheet was originally written by Paolo Besana, and extended by Thomas French and Areti Manataki. Send comments to `A.Manataki@ed.ac.uk`