Informatics 1 - Computation & Logic: Tutorial 3 Solutions

Propositional Logic: Sequent Calculus

Week 5: 14-18 October 2013

Assume the following proof rules, known respectively as '*immediate*', '$\wedge$ *introduction*', '$\rightarrow$ *introduction*', '$\wedge$ *elimination*' and '$\rightarrow$ *elimination*':

$$\frac{\mathcal{A}, X \vdash X}{} \qquad \frac{\mathcal{A} \vdash X \wedge Y}{\begin{array}{c}\mathcal{A} \vdash X \\ \mathcal{A} \vdash Y\end{array}} \qquad \frac{\mathcal{A} \vdash X \rightarrow Y}{\mathcal{A}, X \vdash Y}$$

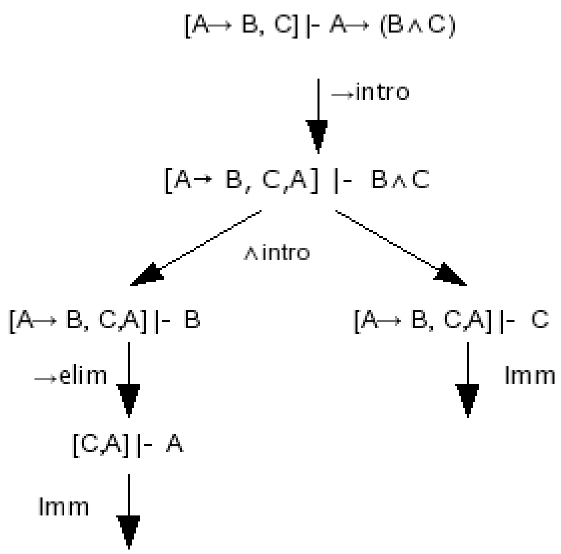
$$\frac{\mathcal{A}, X \wedge Y \vdash Z}{\mathcal{A}, X, Y \vdash Z} \qquad \frac{\mathcal{A}, X \rightarrow Z \vdash Z}{\mathcal{A} \vdash X}$$

Note that $\mathcal{A}$ is a variable over sets of expressions of propositional logic, and X, Y and Z are variables over expressions themselves. A proof rule of the form:

$$\frac{\alpha}{\begin{array}{c}\beta_1 \\ \dots \\ \beta_n\end{array}}$$

means that argument (or sequent) α is valid if all of the arguments $\beta_1, \dots, \beta_n$ are valid. In other words, to prove α you need to prove *all* of $\beta_1, \dots, \beta_n$. Note that it is customary to denote a valid argument using the $\vdash$ symbol to separate premises from conclusion.

For example, using these rules we can prove that an argument like $[A \to B, C] \vdash A \to (B \wedge C)$ is valid, since we are able to cancel all the branches of the proof tree.



Here are some tips for proving the validity of arguments with the use of proof rules:

- All the branches of the proof tree need to be proved.
- A branch of the proof tree is considered to be proved when the *immediate* rule is applied.
- Only one rule can be applied to an argument at a time.
- Remember to include the name of the rule that you are applying at each point.
- If the application of a rule did not help with a proof, cross out the corresponding branch of the tree and try a different rule.

Prove that the following arguments are valid using this method:

1. $[\mathsf{B} \wedge \mathsf{C}] \vdash (\mathsf{A} \rightarrow \mathsf{B}) \wedge (\mathsf{A} \rightarrow \mathsf{C})$

   Applying $\wedge$*intro* we obtain two branches:

   ▷ $[\mathsf{B} \wedge \mathsf{C}] \vdash (\mathsf{A} \rightarrow \mathsf{B})$
   Applying $\rightarrow$ *intro* we obtain:
   $[\mathsf{B} \wedge \mathsf{C}, \mathsf{A}] \vdash \mathsf{B}$
   Applying $\wedge$*elim*:
   $[\mathsf{B}, \mathsf{C}, \mathsf{A}] \vdash \mathsf{B}$
   That can be proved by applying *immediate*.

   ▷ $[\mathsf{B} \wedge \mathsf{C}] \vdash (\mathsf{A} \rightarrow \mathsf{C})$
   Applying $\rightarrow$ *intro* we obtain:
   $[\mathsf{B} \wedge \mathsf{C}, \mathsf{A}] \vdash \mathsf{C}$
   Applying $\wedge$*elim*:
   $[\mathsf{B}, \mathsf{C}, \mathsf{A}] \vdash \mathsf{C}$
   That can be proved by applying *immediate*.

2. $[\mathsf{A} \wedge (\mathsf{B} \wedge \mathsf{C})] \vdash (\mathsf{A} \wedge \mathsf{B}) \wedge \mathsf{C}$

   Using $\wedge$*intro* we obtain two branches:

   ▷ $[\mathsf{A} \wedge (\mathsf{B} \wedge \mathsf{C})] \vdash (\mathsf{A} \wedge \mathsf{B})$
   Applying again $\wedge$*intro* we obtain two sub branches:

   - ▷ $[\mathsf{A} \wedge (\mathsf{B} \wedge \mathsf{C})] \vdash \mathsf{A}$
     Applying $\wedge$*elim* we obtain:
     $[\mathsf{A}, (\mathsf{B} \wedge \mathsf{C})] \vdash \mathsf{A}$
     That can be proved by applying *immediate*.

   - ▷ $[\mathsf{A} \wedge (\mathsf{B} \wedge \mathsf{C})] \vdash \mathsf{B}$
     Applying $\wedge$*elim* we obtain:
     $[\mathsf{A}, \mathsf{B} \wedge \mathsf{C}] \vdash \mathsf{B}$
     Applying $\wedge$*elim* again:
     $[\mathsf{A}, \mathsf{B}, \mathsf{C}] \vdash \mathsf{B}$
     That can be proved by applying *immediate*.

   ▷ $[\mathsf{A} \wedge (\mathsf{B} \wedge \mathsf{C})] \vdash \mathsf{C}$
   Applying $\wedge$*elim* twice we obtain first:
   $[\mathsf{A}, (\mathsf{B} \wedge \mathsf{C})] \vdash \mathsf{C}$
   and then
   $[\mathsf{A}, \mathsf{B}, \mathsf{C}] \vdash \mathsf{C}$
   That can be proved by applying *immediate*.

3. $[\mathsf{A} \to \mathsf{B}, \mathsf{A} \wedge \mathsf{C}] \vdash \mathsf{B} \wedge C$

Using $\wedge$*intro* we obtain two branches:

▷ $[\mathsf{A} \to \mathsf{B}, \mathsf{A} \wedge \mathsf{C}] \vdash \mathsf{B}$
Applying $\to$ *elim* we obtain:
$[\mathsf{A} \wedge \mathsf{C}] \vdash \mathsf{A}$
Applying $\wedge$*elim* we obtain:
$[\mathsf{A}, \mathsf{C}] \vdash \mathsf{A}$
That can be proved by applying *immediate*.

▷ $[\mathsf{A} \to \mathsf{B}, \mathsf{A} \wedge \mathsf{C}] \vdash \mathsf{C}$
Applying $\wedge$*elim* we obtain:
$[\mathsf{A} \to \mathsf{B}, \mathsf{A}, \mathsf{C}] \vdash \mathsf{C}$
That can be proved by applying *immediate*.

Here are some more proof rules, called respectively '$\vee$*introduction left*', '$\vee$*introduction right*', and '$\vee$*elimination'*:

$$\frac{\mathcal{A} \vdash X \vee Y}{\mathcal{A} \vdash X} \qquad \frac{\mathcal{A} \vdash X \vee Y}{\mathcal{A} \vdash Y} \qquad \frac{\mathcal{A},\, X \vee Y \vdash Z}{\begin{array}{c}\mathcal{A}, X \vdash Z \\ \mathcal{A}, Y \vdash Z\end{array}}$$

Prove that the following arguments are valid:

4. $[\mathsf{A} \vee \mathsf{B} \to \mathsf{C},\ \mathsf{C} \to \mathsf{A}] \vdash \mathsf{B} \to \mathsf{C}$

Applying $\to$ *intro* we obtain:
$[\mathsf{A} \vee \mathsf{B} \to \mathsf{C},\ \mathsf{C} \to \mathsf{A},\ \mathsf{B}] \vdash \mathsf{C}$
Applying $\to$ *elim* we obtain:
$[\mathsf{C} \to \mathsf{A},\ \mathsf{B}] \vdash \mathsf{A} \vee \mathsf{B}$
Applying $\vee$*introduction left* we obtain a first branch to prove. If it can be proved, we have finished.

▷ $[\mathsf{C} \to \mathsf{A},\ \mathsf{B}] \vdash \mathsf{A}$
We can try to apply $\to$ *elim* and obtain:
$[\mathsf{B}] \vdash \mathsf{C}$
That cannot be proved. We have to backtrack and try the second branch.

Applying $\vee$*introduction right* we obtain the second branch:

▷ $[\mathsf{C} \to \mathsf{A},\ \mathsf{B}] \vdash \mathsf{B}$
That can be proved by applying *immediate*.

5. $[\mathsf{A} \rightarrow \mathsf{C}] \vdash \mathsf{A} \rightarrow (\mathsf{B} \vee \mathsf{C})$

> Applying $\rightarrow$ *intro* we obtain:
> $[\mathsf{A} \rightarrow \mathsf{C}, \mathsf{A}] \vdash \mathsf{B} \vee \mathsf{C}$
> We first try to apply $\vee introduction\, left$, obtaining:
>
> ▷ $[\mathsf{A} \rightarrow \mathsf{C}, \mathsf{A}] \vdash \mathsf{B}$
> that cannot be satisfied.
>
> Applying $\vee introduction\, right$, we obtain:
>
> ▷ $[\mathsf{A} \rightarrow \mathsf{C}, \mathsf{A}] \vdash \mathsf{C}$
> Applying $\rightarrow$ *elim*, we obtain:
> $[\mathsf{A}] \vdash \mathsf{A}$
> that can be proved by applying *immediate*.

6. Given the above proof rules and some sequent to be proved of the form, $\mathsf{F} \vdash \mathsf{P}$, can you suggest a general proof strategy? (Hint: How did you approach the previous problems?)

> Discussion question: get students to think about a general algorithmic approach or strategy to applying proof rules. Make reference to previous questions.
> As an example, (taken from the notes)
>
> ▷ If P follows directly from the *immediate* rule then it is proved.
>
> ▷ If P is of the form $A \wedge B$ then use $\wedge$*introduction*.
>
> ▷ If P is of the form $A \vee B$ then first try to prove it using $\vee$*introduction left* but, if that fails, then try using $\vee$*introduction right*.
>
> ▷ If P is of the form $A \rightarrow B$ then use $\rightarrow$*introduction*
>
> ▷ Otherwise, apply the $\rightarrow$ *elimination* rule with the first member of F which is of the form $\mathsf{C}_1 \rightarrow \mathsf{P}$. If no proof can be found using this implication statement then take the next statement, $\mathsf{C}_2 \rightarrow \mathsf{P}$, and apply the *implication* rule again. Repeat this procedure until either a proof is found or all the implication statements have been used.

This tutorial exercise sheet was written by Paolo Besana, and extended by Thomas French and Areti Manataki. Send comments to `A.Manataki@ed.ac.uk`