Informatics 1 - Computation & Logic: Tutorial 3

Propositional Logic: Sequent Calculus

Week 5: 14-18 October 2013

Please attempt the entire worksheet in advance of the tutorial, and bring with you all work, including (if a computer is involved) printouts of code and test results. Tutorials cannot function properly unless you do the work in advance.

You may work with others, but you must understand the work; you can't phone a friend during the exam.

Assessment is formative, meaning that marks from coursework do not contribute to the final mark. But coursework is not optional. If you do not do the coursework you are unlikely to pass the exams.

Attendance at tutorials is **obligatory**; please let your tutor know if you cannot attend.

Assume the following proof rules, known respectively as '*immediate*', '$\wedge$ *introduction*', '$\rightarrow$ *introduction*', '$\wedge$ *elimination*' and '$\rightarrow$ *elimination*':

$$\begin{array}{c} \mathcal{A}, X \vdash X \\ \hline \end{array} \qquad \begin{array}{c} \mathcal{A} \vdash X \wedge Y \\ \hline \mathcal{A} \vdash X \\ \mathcal{A} \vdash Y \end{array} \qquad \begin{array}{c} \mathcal{A} \vdash X \rightarrow Y \\ \hline \mathcal{A}, X \vdash Y \end{array}$$

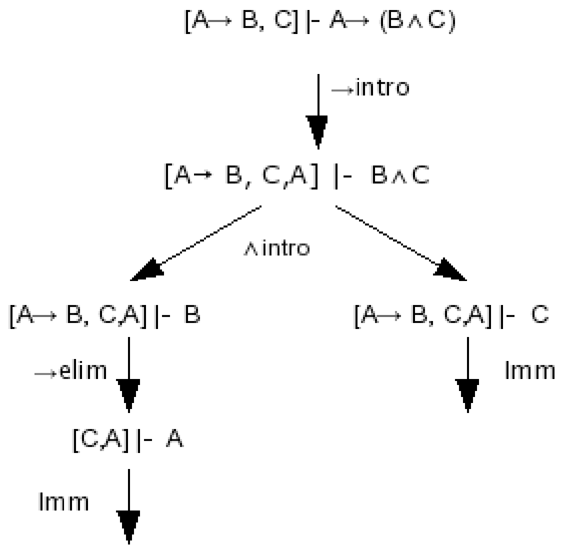
$$\begin{array}{c} \mathcal{A}, X \wedge Y \vdash Z \\ \hline \mathcal{A}, X, Y \vdash Z \end{array} \qquad \begin{array}{c} \mathcal{A}, X \rightarrow Z \vdash Z \\ \hline \mathcal{A} \vdash X \end{array}$$

Note that $\mathcal{A}$ is a variable over sets of expressions of propositional logic, and X, Y and Z are variables over expressions themselves. A proof rule of the form:

$$\begin{array}{c} \alpha \\ \hline \beta_1 \\ \dots \\ \beta_n \end{array}$$

means that argument (or sequent) α is valid if all of the arguments $\beta_1, \dots, \beta_n$ are valid. In other words, to prove α you need to prove *all* of $\beta_1, \dots, \beta_n$. Note that it is customary to denote a valid argument using the $\vdash$ symbol to separate premises from conclusion.

For example, using these rules we can prove that an argument like $[A \rightarrow B, C] \vdash A \rightarrow (B \wedge C)$ is valid, since we are able to cancel all the branches of the proof tree.



Here are some tips for proving the validity of arguments with the use of proof rules:

- All the branches of the proof tree need to be proved.
- A branch of the proof tree is considered to be proved when the *immediate* rule is applied.
- Only one rule can be applied to an argument at a time.
- Remember to include the name of the rule that you are applying at each point.
- If the application of a rule did not help with a proof, cross out the corresponding branch of the tree and try a different rule.

Prove that the following arguments are valid using this method:

1. $[\mathsf{B} \wedge \mathsf{C}] \vdash (\mathsf{A} \rightarrow \mathsf{B}) \wedge (\mathsf{A} \rightarrow \mathsf{C})$

2. $[\mathsf{A}\ \wedge\ (\mathsf{B}\ \wedge\ \mathsf{C})] \vdash (\mathsf{A}\ \wedge\ \mathsf{B})\ \wedge\ \mathsf{C}$

3. $[\mathsf{A} \rightarrow \mathsf{B}, \mathsf{A} \wedge \mathsf{C}] \vdash \mathsf{B} \wedge C$

Here are some more proof rules, called '$\vee$*introduction left*', '$\vee$*introduction right*', and '$\vee$*elimination*', respectively:

$$\frac{\mathcal{A} \vdash X \vee Y}{\mathcal{A} \vdash X} \qquad \frac{\mathcal{A} \vdash X \vee Y}{\mathcal{A} \vdash Y} \qquad \frac{\mathcal{A},\, X \vee Y \vdash Z}{\begin{array}{c}\mathcal{A}, X \vdash Z \\ \mathcal{A}, Y \vdash Z\end{array}}$$

Prove that the following arguments are valid:

4. $[\mathsf{A} \vee \mathsf{B} \rightarrow \mathsf{C},\, \mathsf{C} \rightarrow \mathsf{A}] \vdash \mathsf{B} \rightarrow \mathsf{C}$

5. $[\mathsf{A} \rightarrow \mathsf{C}] \vdash \mathsf{A} \rightarrow (\mathsf{B} \vee \mathsf{C})$

6. Given the above proof rules and some sequent to be proved of the form, $\mathsf{F} \vdash \mathsf{P}$, can you suggest a general proof strategy? (Hint: How did you approach the previous problems?)

This tutorial exercise sheet was written by Paolo Besana, and extended by Thomas French and Areti Manataki. Send comments to `A.Manataki@ed.ac.uk`