

Informatics 1 - Computation and Logic: Tutorial 3 Solutions

Propositional Logic: Sequent Calculus

Week 5: 15 - 19 October 2012

Assume the following proof rules, known respectively as ‘*immediate*’, ‘*∧ introduction*’, ‘*→ introduction*’, ‘*∧ elimination*’ and ‘*→ elimination*’:

$$\frac{\mathcal{A}, X \vdash X}{\mathcal{A}, X \vdash X} \quad \frac{\mathcal{A} \vdash X \wedge Y}{\mathcal{A} \vdash X \quad \mathcal{A} \vdash Y} \quad \frac{\mathcal{A} \vdash X \rightarrow Y}{\mathcal{A}, X \vdash Y}$$
$$\frac{\mathcal{A}, X \wedge Y \vdash Z}{\mathcal{A}, X, Y \vdash Z} \quad \frac{\mathcal{A}, X \rightarrow Z \vdash Z}{\mathcal{A} \vdash X}$$

Note that \mathcal{A} is a variable over sets of expressions of propositional logic, and X , Y and Z are variables over expressions themselves. A proof rule of the form:

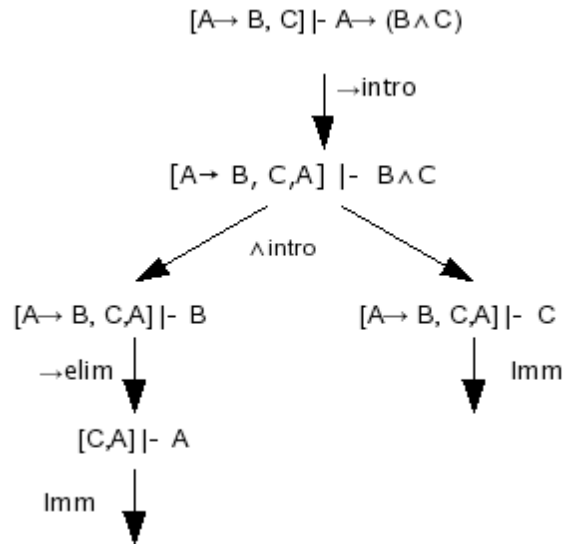
$$\frac{\alpha}{\beta_1}$$

...

$$\beta_n$$

means that argument (or sequent) α is valid if all of the arguments β_1, \dots, β_n are valid. In other words, to prove α you need to prove *all* of β_1, \dots, β_n . Note that, it is customary to denote a valid argument using the \vdash symbol to separate premises from conclusion.

Using these rules we can prove that an argument like $[A \rightarrow B, C] \vdash A \rightarrow (B \wedge C)$ is valid as follows, since we are able to cancel all the branches.



Prove that the following arguments are valid using this method:

- $[B \wedge C] \vdash (A \rightarrow B) \wedge (A \rightarrow C)$

Applying $\wedge\text{intro}$ we obtain two branches:

▷ $[B \wedge C] \vdash (A \rightarrow B)$
 Applying $\rightarrow\text{intro}$ we obtain:
 $[B \wedge C, A] \vdash B$
 Applying $\wedge\text{elim}$:
 $[B, C, A] \vdash B$
 That can be proved immediatly

▷ $[B \wedge C] \vdash (A \rightarrow C)$
 Applying $\rightarrow\text{intro}$ we obtain:
 $[B \wedge C, A] \vdash C$
 Applying $\wedge\text{elim}$:
 $[B, C, A] \vdash C$
 That can be proved immediatly

2. $[A \wedge (B \wedge C)] \vdash (A \wedge B) \wedge C$

Using $\wedge intro$ we obtain two branches:

▷ $[A \wedge (B \wedge C)] \vdash (A \wedge B)$
Applying again $\wedge intro$ we obtain two sub branches:

▷ $[A \wedge (B \wedge C)] \vdash A$
Applying $\wedge elim$ we obtain:
 $[A, (B \wedge C)] \vdash A$
That can be proved immediatly

▷ $[A \wedge (B \wedge C)] \vdash B$
Applying $\wedge elim$ we obtain:
 $[A, B \wedge C] \vdash B$
Applying $\wedge elim$ again:
 $[A, B, C] \vdash B$
That can be proved immediatly

▷ $[A \wedge (B \wedge C)] \vdash C$
Applying $\wedge elim$ twice we obtain first:
 $[A, (B \wedge C)] \vdash C$
and then
 $[A, B, C] \vdash C$
That can be proved immediatly

3. $[A \rightarrow B, A \wedge C] \vdash B \wedge C$

Using $\wedge intro$ we obtain two branches:

▷ $[A \rightarrow B, A \wedge C] \vdash B$
Applying $\rightarrow elim$ we obtain:
 $[A \wedge C] \vdash A$
Applying $\wedge elim$ we obtain:
 $[A, C] \vdash A$
That can be immediatly proved

▷ $[A \rightarrow B, A \wedge C] \vdash C$
Applying $\wedge elim$ we obtain:
 $[A \rightarrow B, A, C] \vdash C$
That can be directly proved

Here are some more proof rules, called respectively '*∨introduction left*', '*∨introduction right*', and '*∨elimination*':

$$\frac{\mathcal{A} \vdash X \vee Y}{\mathcal{A} \vdash X} \qquad \frac{\mathcal{A} \vdash X \vee Y}{\mathcal{A} \vdash Y} \qquad \frac{\mathcal{A}, X \vee Y \vdash Z}{\mathcal{A}, X \vdash Z} \qquad \frac{\mathcal{A}, X \vee Y \vdash Z}{\mathcal{A}, Y \vdash Z}$$

Prove that the following arguments are valid:

4. $[A \vee B \rightarrow C, C \rightarrow A] \vdash B \rightarrow C$

Applying \rightarrow *intro* we obtain:
 $[A \vee B \rightarrow C, C \rightarrow A, B] \vdash C$
 Applying \rightarrow *elim* we obtain:
 $[C \rightarrow A, B] \vdash A \vee B$
 Applying *∨introduction left* we obtain a first branch to prove. If it can be proved, we have finished.

▷
 $[C \rightarrow A, B] \vdash A$
 We can try to apply \rightarrow *elim* and obtain:
 $[B] \vdash C$
 That cannot be proved. We have to backtrack and try the second branch.

Applying *∨introduction right* we obtain the second branch:

▷
 $[C \rightarrow A, B] \vdash B$
 That can be proved immediatly

5. $[A \rightarrow C] \vdash A \rightarrow (B \vee C)$

Applying \rightarrow *intro* we obtain:
 $[A \rightarrow C, A] \vdash B \vee C$
 We first try to apply *∨introduction left*, obtaining:

▷
 $[A \rightarrow C, A] \vdash B$
 that cannot be satisfied.

Applying *∨introduction right*, we obtain:

▷
 $[A \rightarrow C, A] \vdash C$
 Applying \rightarrow *elim*, we obtain:
 $[A] \vdash A$
 that can be proved immediatly

6. Given the above proof rules and some sequent to be proved of the form, $F \vdash P$, can you suggest a general proof strategy? (Hint: How did you approach the previous problems?)

Discussion question: get students to think about a general algorithmic approach or strategy to applying proof rules. Make reference to previous questions.

As an example, (taken from the notes)

- ▷ If P follows directly from the *immediate* rule then it is proved.
- ▷ If P is of the form $A \wedge B$ then use *\wedge introduction*.
- ▷ If P is of the form $A \vee B$ then first try to prove it using *\vee introduction left* but, if that fails, then try using *\vee introduction right*.
- ▷ If P is of the form $A \rightarrow B$ then use *\rightarrow introduction*
- ▷ Otherwise, apply the *\rightarrow elimination* rule with the first member of F which is of the form $C_1 \rightarrow P$. If no proof can be found using this implication statement then take the next statement, $C_2 \rightarrow P$, and apply the *implication* rule again. Repeat this procedure until either a proof is found or all the implication statements have been used.

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