Informatics 1

Lecture 9 Resolution

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$$A \lor \neg A$$

Binary constraints

You may not take both Archeology and Chemistry
If you take Biology you must take Chemistry
You must take Biology or Archeology
If you take Chemistry you must take Divinity
You may not take both Divinity and Biology

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

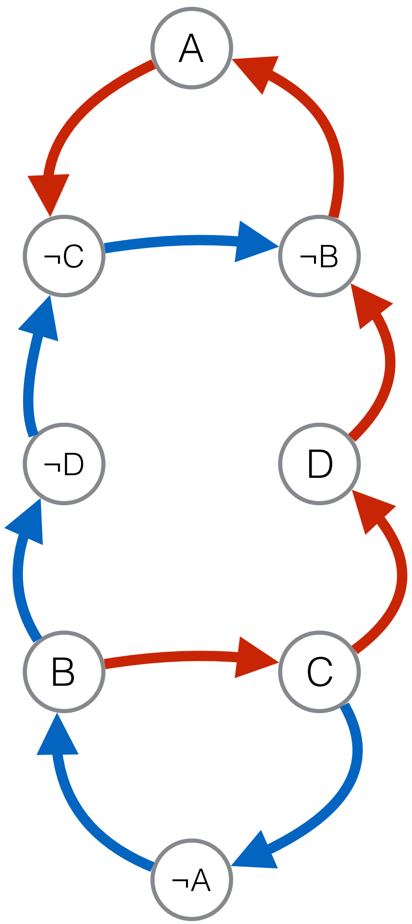
$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

We represent 5 clauses (constraints)
by 10 arrows to give a directed graph.

Here we have 4 atoms A, B, C, D ; so 8 literals.
Any valuation makes 4 literals true; 4 literals false.

The valuation satisfies the constraints
provided no arrow goes from ⊤ to ⊥.

In this case, we can arrange the diagram with a
line that satisfies the arrow rule,
and separates each atom from its negation.



$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (A \vee B) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \to \neg C) \wedge (B \to C) \wedge (\neg B \to A) \wedge (C \to D) \wedge (D \to \neg B)$$

$$\equiv$$

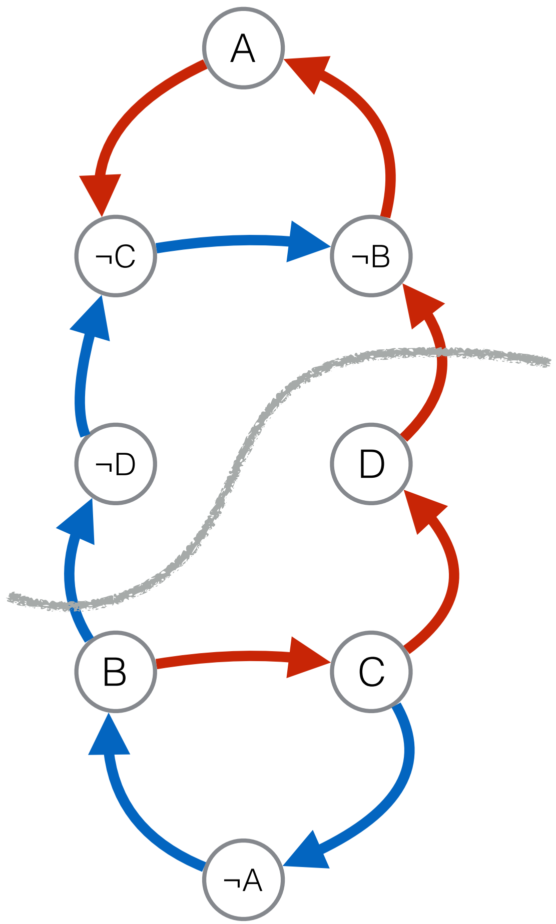
$$(C \to \neg A) \wedge (\neg C \to \neg B) \wedge (\neg A \to B) \wedge (\neg D \to \neg C) \wedge (B \to \neg D)$$

The valuation satisfies the constraints provided no arrow goes from ⊤ to ⊥.

In this case, we can arrange the diagram so we can draw a line that satisfies the arrow rule, and separates each atom from its negation.

For this example, there are only two such lines.

The one shown here makes A true and B, C, D all false



$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (A \vee B) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$\equiv$$

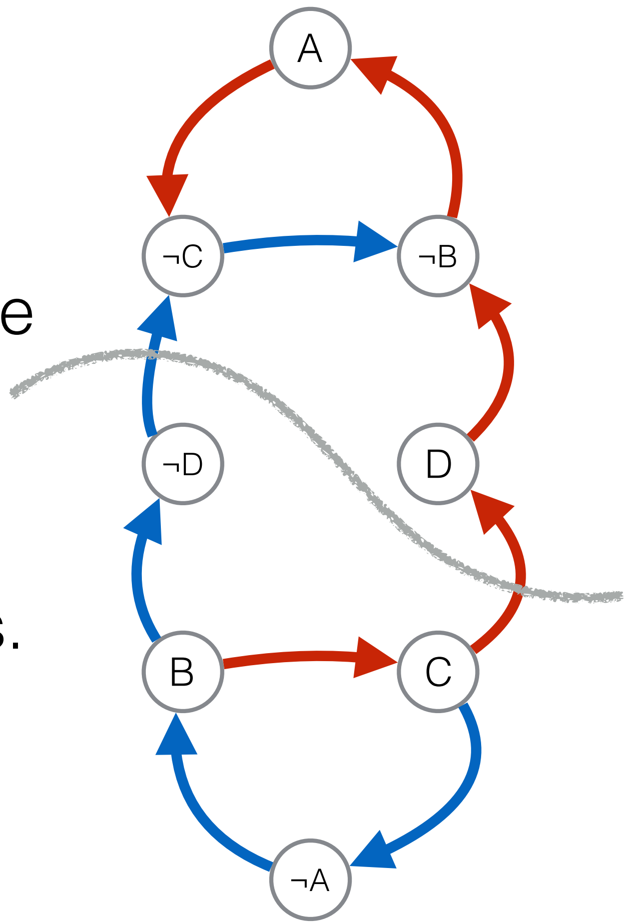
$$(C \rightarrow \neg A) \wedge (\neg C \rightarrow \neg B) \wedge (\neg A \rightarrow B) \wedge (\neg D \rightarrow \neg C) \wedge (B \rightarrow \neg D)$$

The valuation satisfies the constraints provided no arrow goes from ⊤ to ⊥.

In this case, we can arrange the diagram so we can draw a line that satisfies the arrow rule, and separates each atom from its negation.

For this example, there are only two such lines.

The one shown here makes A, D true and B, C false



$$(\neg A \lor \neg C) \land (\neg B \lor C) \land (A \lor B) \land (\neg C \lor D) \land (\neg D \lor \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \land (B \rightarrow C) \land (\neg B \rightarrow A) \land (C \rightarrow D) \land (D \rightarrow \neg B)$$

$$\equiv$$

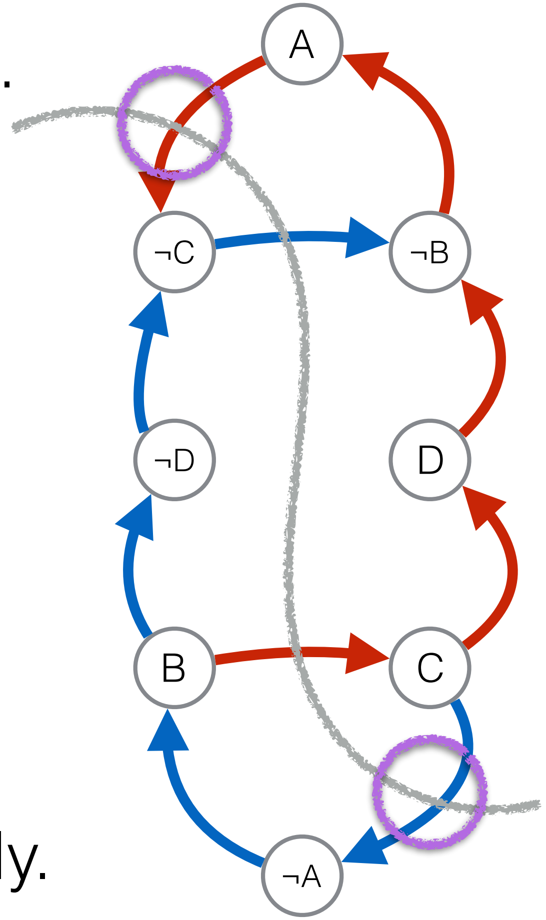
$$(C \rightarrow \neg A) \land (\neg C \rightarrow \neg B) \land (\neg A \rightarrow B) \land (\neg D \rightarrow \neg C) \land (B \rightarrow \neg D)$$

For this example, there are only two such lines.

For example, the valuation that makes
A, C, D true and B false
violates the arrow rule

Our analysis shows that our constraints are
equivalent to the requirement
$A \wedge \neg B \wedge \neg C$

We will find a way to compute this result logically.



$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (A \vee B) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

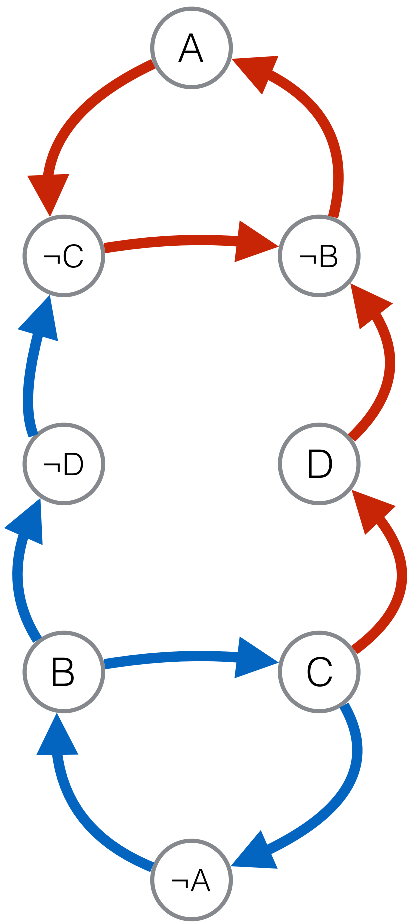
$$\equiv$$

$$(C \rightarrow \neg A) \wedge (\neg C \rightarrow \neg B) \wedge (\neg A \rightarrow B) \wedge (\neg D \rightarrow \neg C) \wedge (B \rightarrow \neg D)$$

Our analysis shows that our constraints are equivalent to the requirement

$$A \wedge \neg B \wedge \neg C$$

Two expressions are equivalent iff they are satisfied by the same valuations.



$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$=$$

$$(C \rightarrow \neg A) \wedge (B \rightarrow C) \wedge (B \rightarrow C)$$

$$=$$

$$A \wedge \neg B \wedge \neg C$$

0	0	0	0
1	1	1	1
1	1	1	1
1	1	1	1

$(R \vee B)$

∧

1	1	1	0
1	1	1	0
1	1	1	0
1	1	1	0

$(\neg A \vee G)$

=

0	0	0	0
1	1	1	0
1	1	1	0
1	1	1	0

$(R \vee B) \wedge (\neg A \vee G)$

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1	1	1	1
1	1	1	1
0	0	1	1
0	0	1	1

$(\neg R \vee A)$

∧

1	1	1	1
0	1	1	0
0	1	1	0
1	1	1	1

$(\neg B \vee G)$

=

1	1	1	1
0	1	1	0
0	0	1	0
0	0	1	1

$(\neg R \vee A) \wedge (\neg B \vee G)$

0	0	0	0
1	1	1	1
1	1	1	1
1	1	1	1

$(R \vee B)$

$\wedge$

1	1	1	0
1	1	1	0
1	1	1	0
1	1	1	0

$(\neg A \vee G)$

$=$

0	0	0	0
1	1	1	0
1	1	1	0
1	1	1	0

$(R \vee B) \wedge (\neg A \vee G)$

1	1	1	1
1	1	1	1
0	0	1	1
0	0	1	1

$(\neg R \vee A)$

$\wedge$

1	1	1	1
0	1	1	0
0	1	1	0
1	1	1	1

$(\neg B \vee G)$

$=$

1	1	1	1
0	1	1	0
0	0	1	0
0	0	1	1

$(\neg R \vee A) \wedge (\neg B \vee G)$

0	0	0	0
1	1	1	1
1	1	1	1
1	1	1	1

(R ∨ B)

∧

1	1	1	0
1	1	1	0
1	1	1	0
1	1	1	0

(¬A ∨ G)

=

0	0	0	0
1	1	1	0
1	1	1	0
1	1	1	0

(R ∨ B)∧(¬A ∨ G)

1	1	1	1
1	1	1	1
0	0	1	1
0	0	1	1

(¬R ∨ A)

∧

1	1	1	1
0	1	1	0
0	1	1	0
1	1	1	1

(¬B ∨ G)

=

1	1	1	1
0	1	1	0
0	0	1	0
0	0	1	1

(¬R ∨ A)∧(¬B ∨ G)

0	0	0	0
1	1	1	1
1	1	1	1
1	1	1	1

(R ∨ B)

∧

1	1	1	0
1	1	1	0
1	1	1	0
1	1	1	0

(¬A ∨ G)

=

0	0	0	0
1	1	1	0
1	1	1	0
1	1	1	0

(R ∨ B)∧(¬A ∨ G)

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1	1	1	1
1	1	1	1
0	0	1	1
0	0	1	1

(¬R ∨ A)

∧

1	1	1	1
0	1	1	0
0	1	1	0
1	1	1	1

(¬B ∨ G)

=

1	1	1	1
0	1	1	0
0	0	1	0
0	0	1	1

(¬R ∨ A)∧(¬B ∨ G)

A G B R

0	0	0	0
1	1	1	0
1	1	1	0
1	1	1	0

A G B R

1	1	1	1
0	1	1	0
0	0	1	0
0	0	1	1

$$\big((R \vee B)\wedge(\neg A \vee G)\big) \bigvee \big((\neg R \vee A)\wedge(\neg B \vee G)\big)$$

$$= \begin{matrix} (R \vee B \vee \neg R \vee A) \\ \wedge \\ (R \vee B \vee \neg B \vee G) \\ \wedge \\ (\neg A \vee G \vee \neg R \vee A) \\ \wedge \\ (\neg A \vee G \vee \neg B \vee G) \end{matrix} =$$

A G B R

1	1	1	1
1	1	1	0
1	1	1	0
1	1	1	1

$$(\neg A \vee G \vee \neg B)$$

$$(a+b)\times(c+d) = (a\times c)+(a\times d)+(b\times c)+(b\times d)$$

$$(a\wedge b)\vee\ (c\wedge d) = (a\vee c)\ \wedge\ (a\vee d)\ \wedge (b\vee c)\ \wedge\ (b\vee d)$$

$$\begin{aligned}\Gamma &= (R\vee B)\wedge(\neg A\vee G)\\ &= \qquad a \qquad \wedge \qquad b\\ \Delta &= (\neg R\vee A)\wedge(\neg B\vee G)\\ &= \qquad c \qquad \wedge \qquad d\end{aligned}$$

$$\begin{aligned}\Gamma\vee\Delta = \quad & (R\vee B\vee\neg R\vee A) && (\text{trivial } a\vee c)\\ & \wedge(R\vee B\vee\neg B\vee G) && (\text{trivial } a\vee d)\\ & \wedge(\neg A\vee G\vee\neg R\vee A) && (\text{trivial } b\vee c)\\ & \wedge(\neg A\vee G\vee\neg B\vee G) && (b\vee d)\end{aligned}$$

$$\Gamma\vee\Delta = \neg A\vee G\vee\neg B$$

Combining constraints disjunction

$$\Gamma = \quad (R \vee B) \wedge (\neg A \vee G)$$

$$\Delta = (\neg R \vee A) \wedge (\neg B \vee G)$$

$$\begin{aligned} \Gamma \vee \Delta = \quad & (R \vee B \vee \neg R \vee A) && \text{(trivial)} \\ & \wedge (R \vee B \vee \neg B \vee G) && \text{(trivial)} \\ & \wedge (\neg A \vee G \vee \neg R \vee A) && \text{(trivial)} \\ & \wedge (\neg A \vee G \vee \neg B \vee G) && \end{aligned}$$

$\Gamma \vee \Delta = \neg A \vee G \vee \neg B$ Either Γ or Δ or both

Combining contraints control

$$(X \vee \Gamma) \wedge (\neg X \vee \Delta)$$

If X then Δ else Γ

$$(X ? \Delta : \Gamma)$$

Combining constraints

$$(X \vee \Gamma) \wedge (\neg X \vee \Delta)$$

$$\Gamma = (R \vee B) \wedge (\neg A \vee G)$$
$$\Delta = (\neg R \vee A) \wedge (\neg B \vee G)$$

$$X \vee \Gamma = (X \vee R \vee B) \wedge (X \vee \neg A \vee G)$$
$$\neg X \vee \Delta = (\neg X \vee \neg R \vee A) \wedge (\neg X \vee \neg B \vee G)$$

Combining constraints

$$
\begin{aligned}
(X \vee \Gamma) \wedge (\neg X \vee \Delta) = \quad & (X \vee R \vee B) \\
& \wedge (X \vee \neg A \vee G) \\
& \wedge (\neg X \vee \neg R \vee A) \\
& \wedge (\neg X \vee \neg B \vee G)
\end{aligned}
$$

To solve these constraints …

if we are free to choose X

first solve Γ ∨ Δ

then choose X

Combining constraints

$$(X \vee \Gamma) \wedge (\neg X \vee \Delta) = \quad (X \vee R \vee B) \wedge (X \vee \neg A \vee G) \wedge (\neg X \vee \neg R \vee A) \wedge (\neg X \vee \neg B \vee G)$$

Γ ∨ Δ

is easy to construct

$$\Gamma \vee \Delta = \quad (R \vee B \vee \neg R \vee A) \wedge (R \vee B \vee \neg B \vee G) \wedge (\neg A \vee G \vee \neg R \vee A) \wedge (\neg A \vee G \vee \neg B \vee G)$$

$$\Gamma \vee \Delta = \neg A \vee G \vee \neg B$$

We pair

each constraint in Γ

with

each constraint in Δ

Combining constraints

$$(X \vee \Gamma) \wedge (\neg X \vee \Delta) = \quad (X \vee R \vee B)$$
$$\wedge\, (X \vee \neg A \vee G)$$
$$\wedge\, (\neg X \vee \neg R \vee A)$$
$$\wedge\, (\neg X \vee \neg B \vee G)$$

$$\Gamma \vee \Delta = \quad (R \vee B \vee \neg R \vee A)$$
$$\wedge\, (R \vee B \vee \neg B \vee G)$$
$$\wedge\, (\neg A \vee G \vee \neg R \vee A)$$
$$\wedge\, (\neg A \vee G \vee \neg B \vee G)$$

$$\Gamma \vee \Delta = \neg A \vee G \vee \neg B$$

Choose a solution e.g. R ∧ B ∧ ¬A ∧ G
then choose X ….

Combining constraints

$$(X \vee \Gamma) \wedge (\neg X \vee \Delta) = \quad (X \vee R \vee B)$$
$$\wedge\, (X \vee \neg A \vee G)$$
$$\wedge\, (\neg X \vee \neg R \vee A)$$
$$\wedge\, (\neg X \vee \neg B \vee G)$$

$$\Gamma \vee \Delta = \neg A \vee G \vee \neg B$$

Choose a solution
e.g. R ∧ B ∧ ¬A ∧ G
then choose X ....

R ∧ B ∧ ¬A ∧ G ⊨ Γ   so we choose to make X false
R ∧ B ∧ ¬A ∧ G ∧ ¬X ⊨ ¬X
so : R ∧ B ∧ ¬A ∧ G ∧ ¬X ⊨ ¬X ∨ Δ
but: R ∧ B ∧ ¬A ∧ G ∧ ¬X ⊨ X ∨ Γ (since it satisfies Γ)

topsy-turvy

We run this idea backwards to simplify and solve an arbitrary set of constraints

$$A \quad B \quad C \quad D \quad E \quad F$$

$$\neg B \vee \neg D \vee \neg F$$

$$\neg B \vee \neg D \vee F$$

$$\neg B \vee D \vee \neg F$$

$$B \vee \neg D \vee \neg F$$

$$A \vee C \vee \neg E$$

$$\neg C \vee D \vee F$$

$$A \vee B \vee D$$

$$\neg D \vee E \vee F$$

$$\neg A \vee \neg B \vee C$$

$$\neg C \vee \neg D \vee F$$

topsy-turvy

We run this idea backwards to simplify and solve an arbitrary set of constraints

$$A \; B \; C \; D \; E \; F$$

reorder to bring constraints mentioning A together

$$A \lor C \lor \neg E$$

$$A \lor B \lor D$$

$$\neg A \lor \neg B \lor C$$

$$\neg B \lor \neg D \lor \neg F$$

$$\neg B \lor \neg D \lor F$$

$$\neg B \lor D \lor \neg F$$

$$B \lor \neg D \lor \neg F$$

$$\neg C \lor D \lor F$$

$$\neg D \lor E \lor F$$

$$\neg C \lor \neg D \lor F$$

$A \vee C \vee \neg E$	$A \vee ((C \vee \neg E) \wedge (B \vee D))$
$A \vee B \vee D$	$\neg A \vee \neg B \vee C$
$\neg A \vee \neg B \vee C$	
$\neg B \vee \neg D \vee \neg F$	$\neg B \vee \neg D \vee \neg F$
$\neg B \vee \neg D \vee F$	$\neg B \vee \neg D \vee F$
$\neg B \vee D \vee \neg F$	$\neg B \vee D \vee \neg F$
$B \vee \neg D \vee \neg F$	$B \vee \neg D \vee \neg F$
$\neg C \vee D \vee F$	$\neg C \vee D \vee F$
$\neg D \vee E \vee F$	$\neg D \vee E \vee F$
$\neg C \vee \neg D \vee F$	$\neg C \vee \neg D \vee F$

$$A \vee ((C \vee \neg E) \wedge (B \vee D))$$

$$\neg A \vee \neg B \vee C$$

$$=$$

$$(A \vee \Gamma) \wedge (\neg A \vee \mathbf{\Delta})$$

$$\Gamma = (C \vee \neg E) \wedge (B \vee D)$$

$$\mathbf{\Delta} = (\neg B \vee C)$$

$$\Gamma \vee \mathbf{\Delta} = (C \vee \neg E \vee \neg B \vee C) \wedge (B \vee D \vee \neg B \vee C)$$

$$= C \vee \neg E \vee \neg B$$

if we can satisfy this, together with the remaining constraints
Then we can find a value for A

$$A \lor C \lor \neg E$$

$$A \lor B \lor D$$

$$\neg A \lor \neg B \lor C$$

$$\neg B \lor \neg D \lor \neg F$$

$$\neg B \lor \neg D \lor F$$

$$\neg B \lor D \lor \neg F$$

$$B \lor \neg D \lor \neg F$$

$$\neg C \lor D \lor F$$

$$\neg D \lor E \lor F$$

$$\neg C \lor \neg D \lor F$$

$$A \lor ((C \lor \neg E) \land (B \lor D))$$

$$\neg A \lor \neg B \lor C$$

$$\neg B \lor \neg D \lor \neg F$$

$$\neg B \lor \neg D \lor F$$

$$\neg B \lor D \lor \neg F$$

$$B \lor \neg D \lor \neg F$$

$$\neg C \lor D \lor F$$

$$\neg D \lor E \lor F$$

$$\neg C \lor \neg D \lor F$$

$A \lor C \lor \neg E$

$A \lor B \lor D$

$\neg A \lor \neg B \lor C$

$\neg B \lor \neg D \lor \neg F$

$\neg B \lor \neg D \lor F$

$\neg B \lor D \lor \neg F$

$B \lor \neg D \lor \neg F$

$\neg C \lor D \lor F$

$\neg D \lor E \lor F$

$\neg C \lor \neg D \lor F$

$\neg B \lor C \lor \neg E$

$\neg B \lor \neg D \lor \neg F$

$\neg B \lor \neg D \lor F$

$\neg B \lor D \lor \neg F$

$B \lor \neg D \lor \neg F$

$\neg C \lor D \lor F$

$\neg D \lor E \lor F$

$\neg C \lor \neg D \lor F$

$$\neg B \vee C \vee \neg E$$

$$\neg B \vee \neg D \vee \neg F$$

$$\neg B \vee \neg D \vee F$$

$$\neg B \vee D \vee \neg F$$

$$B \vee \neg D \vee \neg F$$

$$C \vee \neg E \vee \neg D \vee \neg F$$

$$\neg D \vee \neg F \vee \neg D \vee \neg F$$

$$\neg D \vee F \vee \neg D \vee \neg F$$

$$D \vee \neg F \vee \neg D \vee \neg F$$

$$C \vee \neg E \vee \neg D \vee \neg F$$

$$\neg D \vee \neg F \vee \neg D \vee \neg F$$

$$\neg D \vee F \vee \neg D \vee \neg F$$

$$D \vee \neg F \vee \neg D \vee \neg F$$

$$= C \vee \neg E \vee \neg D \vee \neg F$$

$$\neg D \vee \neg F$$

$$= \neg D \vee \neg F$$

$$A \lor C \lor \neg E$$

$$A \lor B \lor D$$

$$\neg A \lor \neg B \lor C$$

$$\neg B \lor \neg D \lor \neg F$$

$$\neg B \lor \neg D \lor F$$

$$\neg B \lor D \lor \neg F$$

$$B \lor \neg D \lor \neg F$$

$$\neg C \lor D \lor F$$

$$\neg D \lor E \lor F$$

$$\neg C \lor \neg D \lor F$$

$$\neg D \lor \neg F$$

$$\neg C \lor D \lor F$$

$$\neg D \lor E \lor F$$

$$\neg C \lor \neg D \lor F$$

Make
C false
D false

E F can be chosen freely
make both true
Make B false; A true