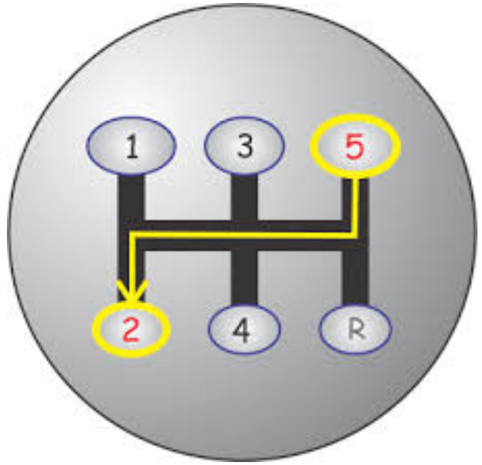
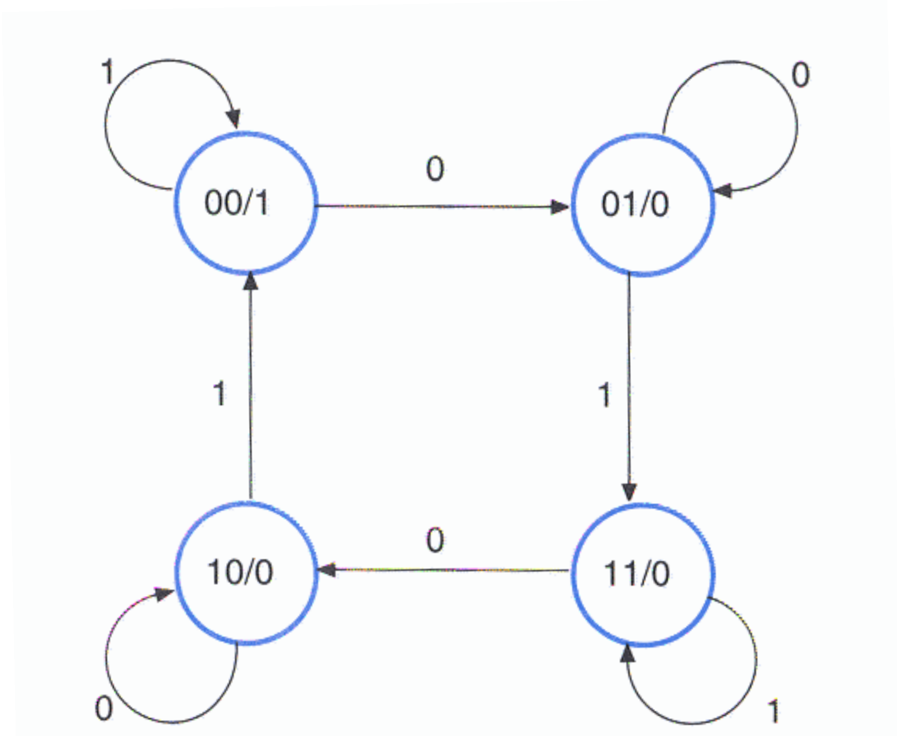
Informatics 1

Lecture 10 All Change



Michael Fourman

Present State A B	Input X	Next State A B	Output Y
0 0	0	0 0	0
0 0	1	0 1	0
0 1	0	0 0	1
0 1	1	1 1	0
1 0	0	0 0	1
1 0	1	1 0	0
1 1	0	0 0	1
1 1	1	1 0	0



$$A \lor C \lor \neg E$$

$$A \lor B \lor D$$

$$\neg A \lor \neg B \lor C$$

$$\neg B \lor \neg D \lor \neg F$$

$$\neg B \lor \neg D \lor F$$

$$\neg B \lor D \lor \neg F$$

$$B \lor \neg D \lor \neg F$$

$$\neg C \lor D \lor F$$

$$\neg D \lor E \lor F$$

$$\neg C \lor \neg D \lor F$$

$$\neg D \lor \neg F$$

$$\neg C \lor D \lor F$$

$$\neg D \lor E \lor F$$

$$\neg C \lor \neg D \lor F$$

Make
C false
D false

E F can be chosen freely
make both true
Make B false; A true

A	
$A \vee C \vee \neg E$	
$A \vee B \vee D$	
$\neg A \vee \neg B \vee C$	
$\neg B \vee \neg D \vee \neg F$	
$\neg B \vee \neg D \vee F$	
$\neg B \vee D \vee \neg F$	
$B \vee \neg D \vee \neg F$	
$\neg C \vee D \vee F$	
$\neg D \vee E \vee F$	
$\neg C \vee \neg D \vee F$	

A

$$(A \vee \Gamma) \wedge (\neg A \vee \Delta) \wedge \Omega$$

$$A \vee C \vee \neg E$$
$$A \vee B \vee D$$
$$\neg A \vee \neg B \vee C$$
$$\neg B \vee \neg D \vee \neg F$$
$$\neg B \vee \neg D \vee F$$
$$\neg B \vee D \vee \neg F$$
$$B \vee \neg D \vee \neg F$$
$$\neg C \vee D \vee F$$
$$\neg D \vee E \vee F$$
$$\neg C \vee \neg D \vee F$$

$$(\Gamma \vee \Delta) \wedge \Omega$$

A

$A \vee C \vee \neg E$	$C \vee \neg E \vee \neg B \vee C$
$A \vee B \vee D$	$\cancel{B \vee D \vee \neg B \vee C}$
$\neg A \vee \neg B \vee C$	
$\neg B \vee \neg D \vee \neg F$	
$\neg B \vee \neg D \vee F$	
$\neg B \vee D \vee \neg F$	
$B \vee \neg D \vee \neg F$	
$\neg C \vee D \vee F$	
$\neg D \vee E \vee F$	
$\neg C \vee \neg D \vee F$	

A

$^{A}A \vee C \vee \neg E$	$C \vee \neg E \vee \neg B \vee C$
$^{A}A \vee B \vee D$	$\cancel{B \vee D \vee \neg B \vee C}$
$^{A}\neg A \vee \neg B \vee C$	
$\neg B \vee \neg D \vee \neg F$	
$\neg B \vee \neg D \vee F$	
$\neg B \vee D \vee \neg F$	
$B \vee \neg D \vee \neg F$	
$\neg C \vee D \vee F$	
$\neg D \vee E \vee F$	
$\neg C \vee \neg D \vee F$	

A	C	
${}^{A}A \lor C \lor \neg E$	${}^{C}C \lor \neg E \lor \neg B$	$D \lor F \lor \neg E \lor \neg B$
${}^{A}A \lor B \lor D$	$\cancel{B \lor D \lor \neg B \lor C}$	$\neg D \lor F \lor \neg E \lor \neg B$
${}^{A}\neg A \lor \neg B \lor C$		
$\neg B \lor \neg D \lor \neg F$		
$\neg B \lor \neg D \lor F$		
$\neg B \lor D \lor \neg F$		
$B \lor \neg D \lor \neg F$		
${}^{C}\neg C \lor D \lor F$		
$\neg D \lor E \lor F$		
${}^{C}\neg C \lor \neg D \lor F$		

When does resolution stop?
What does a set of clauses look like when there are no opportunities for resolution?

If we produce the empty clause {} by resolving {X} and {¬X} then the constraints are not satisfiable.

If resolution stops without producing the empty clause, then every remaining literal is pure – its negation does not appear.

So, we can construct a satisfying valuation.

This shows that the resolution procedure is **complete** – if a set of constraints is inconsistent we will produce the empty clause. Otherwise we can produce a satisfying valuation.

To produce conjunctive normal form (CNF)

eliminate $\leftrightarrow$ $\rightarrow$

push negations in

push $\vee$ inside $\wedge$

$\neg(a \rightarrow b) = a \wedge \neg b$ $\quad a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$ $\quad a \rightarrow b = \neg a \vee b$

$\neg(a \vee b) = \neg a \wedge \neg b$ $\quad \neg(a \vee b) = \neg a \wedge \neg b$

$\neg 0 = 1$ $\quad \neg\neg a = a$ $\quad \neg 1 = 0$

$a \vee 1 = 1$ $\quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ $\quad a \wedge 0 = 0$

$a \vee 0 = a$ $\quad a \vee \neg a = 1$ $\quad a \wedge \neg a = 0$ $\quad a \wedge 1 = a$

A farmer has to get a wolf, a goose, and a sack of corn across a river.

She has a boat, which can only carry her and one other thing.

If the wolf and the goose are left together, the wolf will eat the goose.

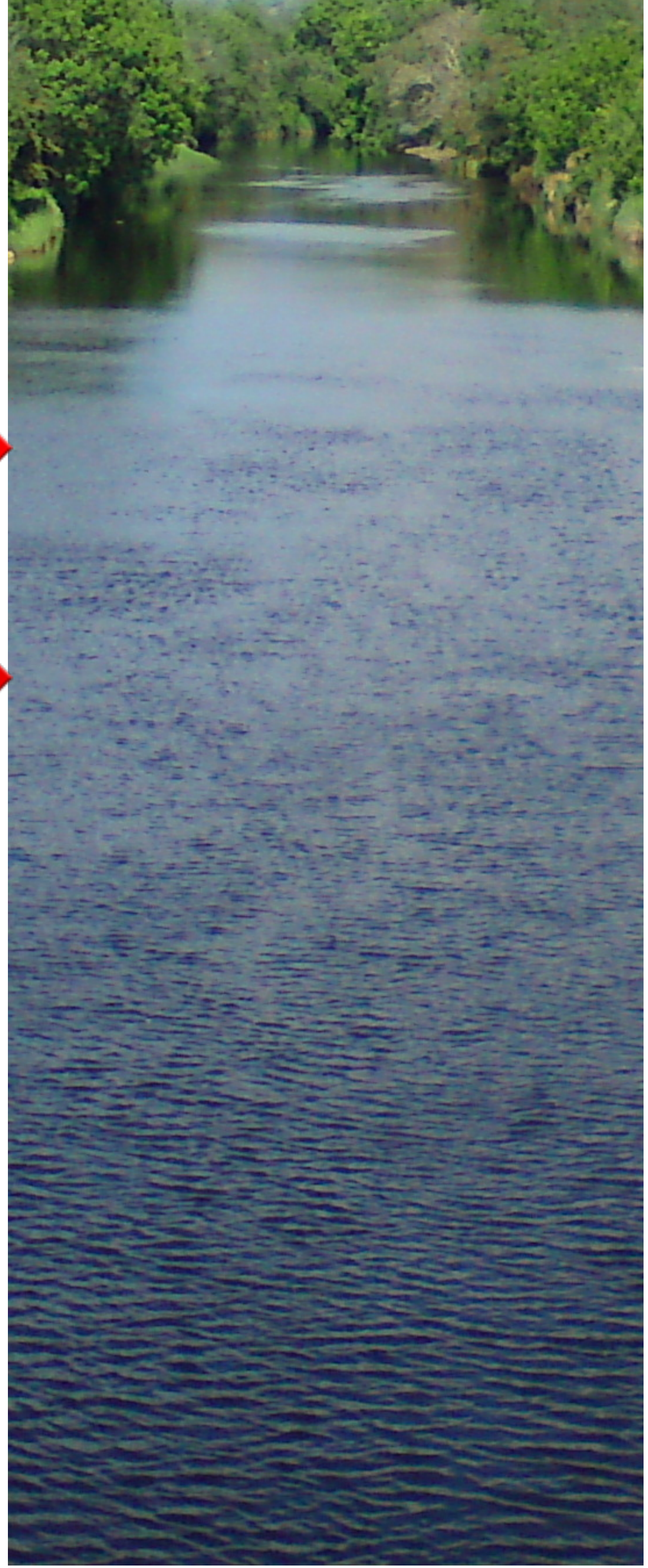
If the goose and the corn are left together, the goose will eat the corn.

How does she do it?

WW

GW

CW



FE

WW

CW

FE

GE

WW

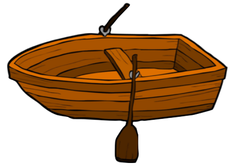
GE

CW

FB

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

We have a dozen propositions.
Each proposition may be true or false.
Each combination of truth values defines a state of the system.

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

These 12 propositions allow $4096 = 2^{12}$ states.

Some of these are impossible - each thing can only be in one place at a time.
There only 81 possible states. How do we arrive at this number?

How can we use logic to specify the possible states?

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

Some of the 81 possible states are not legal.
The farmer can only take one load in the boat.

How many of the possible states have at most the farmer and one load in the boat?

How can we use logic to specify the legal states?

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

Some of the legal, possible states are not safe.
The farmer cannot safely leave the wolf with the goose or the goose with the corn.

How many of the legal, possible states are safe?

How can we use logic to specify the safe states?

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

Once you have identified the safe, legal, possible states,
you can draw a diagram showing the possible transitions from one state to another.

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

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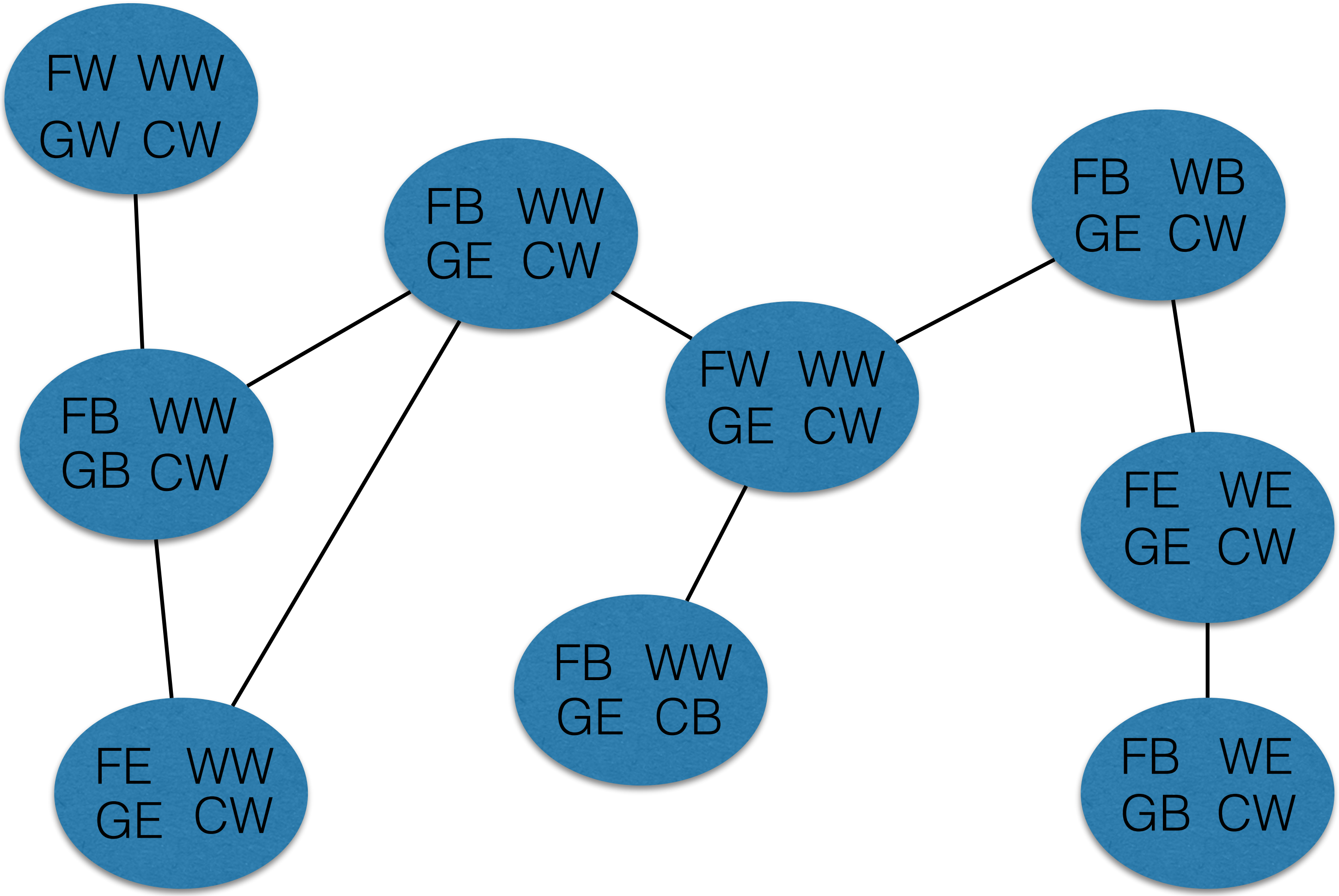
She has a boat, which can only carry her and one other thing.

If the wolf and the goose are left together, the wolf will eat the goose.

If the goose and the corn are left together, the chicken will eat the corn.



A farmer has to get a wolf, a goose, and a sack of corn across a river.

How can we use logic to specify the transitions?

The

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The jealous husbands and The missionaries and cannibals

IAN PRESSMAN AND DAVID SINGMASTER

The classical river crossing problem of the jealous husbands involves three couples who have to cross a river using a boat that holds just two people. The jealousy of the husbands requires that no wife can be in the presence of another man without her husband being present. This can be accomplished in 11 crossings (i.e. one-way trips). Tartaglia gave a sketchy solution for four couples but Bachet pointed out that this was erroneous and that four couples could not get across the river. In 1879, De Fontenay pointed out that four or more couples could cross the river if there was an island in the river and gave a solution for n couples in $8n - 8$ crossings. Dudeney improved the solution for $n = 4$ and Ball noted that this gives $6n - 7$ crossings for n couples.

From the results of a computer search, we have discovered solutions in 16 crossings for $n = 4$ and in $4n + 1$ crossings for $n > 4$ and we have proven that these are the minimal number of crossings. We have also found that De Fontenay's solution should be in $8n - 6$ crossings and that this is the minimal number of crossings when trips from bank to bank are prohibited.

The more recent missionaries and cannibals problem has n of each type of person and the conditions are that the cannibals must never outnumber the missionaries at any location. This is a proper weakening of the jealous husbands problem. When bank-to-bank crossings are prohibited, De Fontenay's method already uses the least possible number of crossings, even disregarding any conditions, hence is also optimal for this version of the problem. When bank-to-bank crossings are permitted, the 16 crossing solution for the jealous husbands can be reduced to 15 and this generates a solution in $4n - 1$ crossings, which is the minimal number of crossings for $n \geqslant 3$.

How can we use propositional logic to model the jealous husbands problem?

How many legal safe states are there for this problem?

Can we use propositional logic to model the missionaries and cannibals problem?

A River-Crossing Problem in Cross-Cultural Perspective

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1. Introduction Most mathematicians react with interest to the challenge of a logical puzzle. In fact, some story puzzles have become such favorites that many of us cannot even recall where we learned them. Perhaps one of the best known is the puzzle in which a man must ferry across a river a wolf, a goat, and a head of cabbage. The difficulty is that the available boat can only carry him and one other thing but neither the wolf and goat nor goat and cabbage can be left alone together. Story puzzles are simple and accessible because they do not rely on any particular body of knowledge and yet they are mathematical in that a stated goal must be achieved under a given set of logical constraints. Attention to logic, as evidenced by the existence of these puzzles, is not the exclusive province of any one culture or subculture. Here, the river-crossing problem, in African cultures as well as in Western culture, will be used as an explicit example of the panhuman concern for mathematical ideas. Story puzzles are expressions of their cultures and so variations will be seen in the characters, the settings and the way in which the logical problem is framed.

2. Western versions The Western origin of the wolf, goat, and cabbage puzzle is most often attributed to a set of 53 problems designed to challenge youthful minds, "Propositiones and acuendos iuvenes." Although circulated around the year 1000, Alcuin of York (735–804) is said to have authored these as he referred to them in a letter to his most famous student, Charlemagne. The solution given by these works is to carry over the goat, then transport the wolf and return with the goat, then carry over the cabbage, then carry over the goat. A second solution, which simply interchanges the wolf and cabbage, is often attributed to the French mathematician Chuquet in 1484 but is found even earlier in the twelfth century in Germany in the succinct form of Latin hexameter [**1**, **4**, **5**, **23**].

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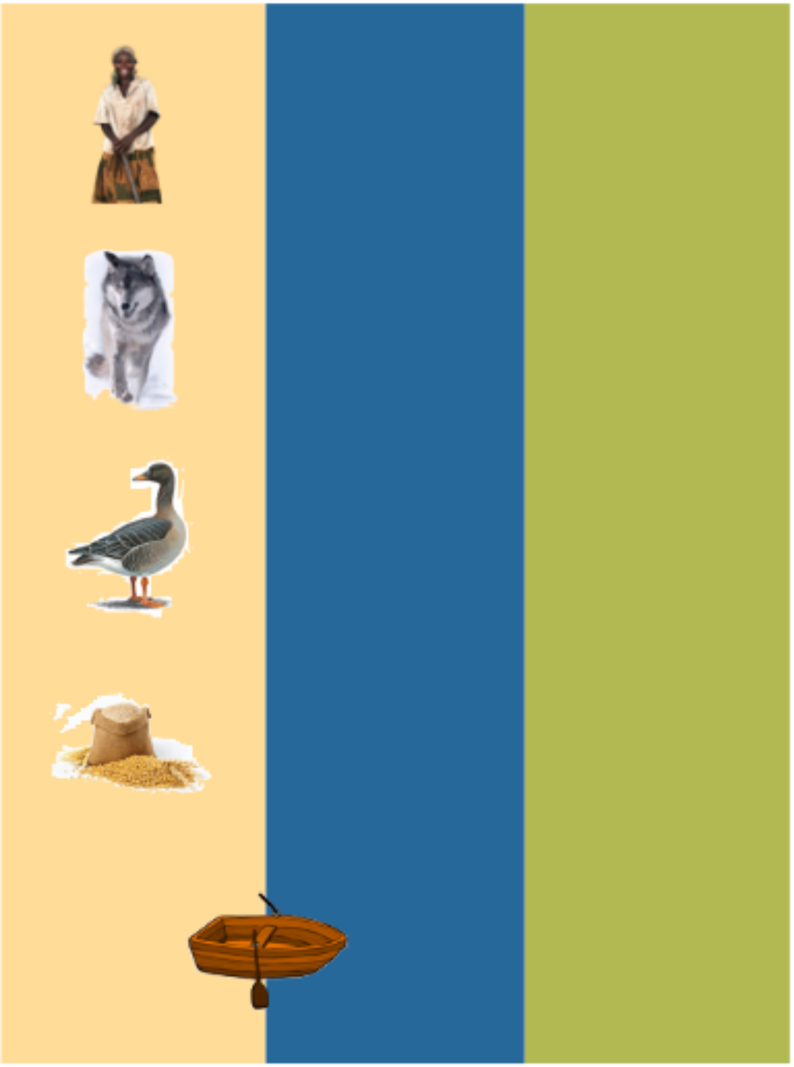

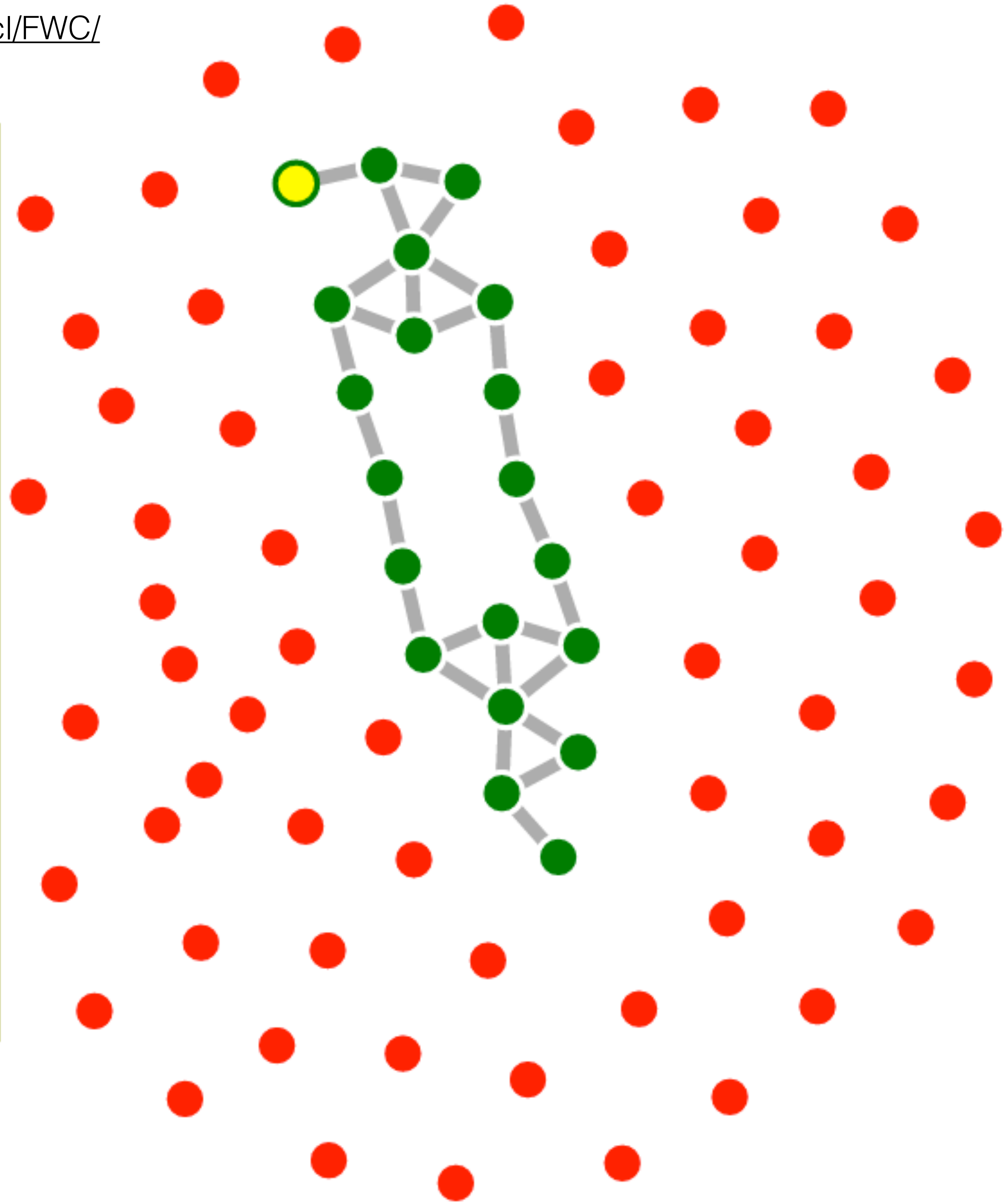

A farmer has to get a wolf, a goose, and a sack of corn across a river.

How can we use logic to specify the transitions?

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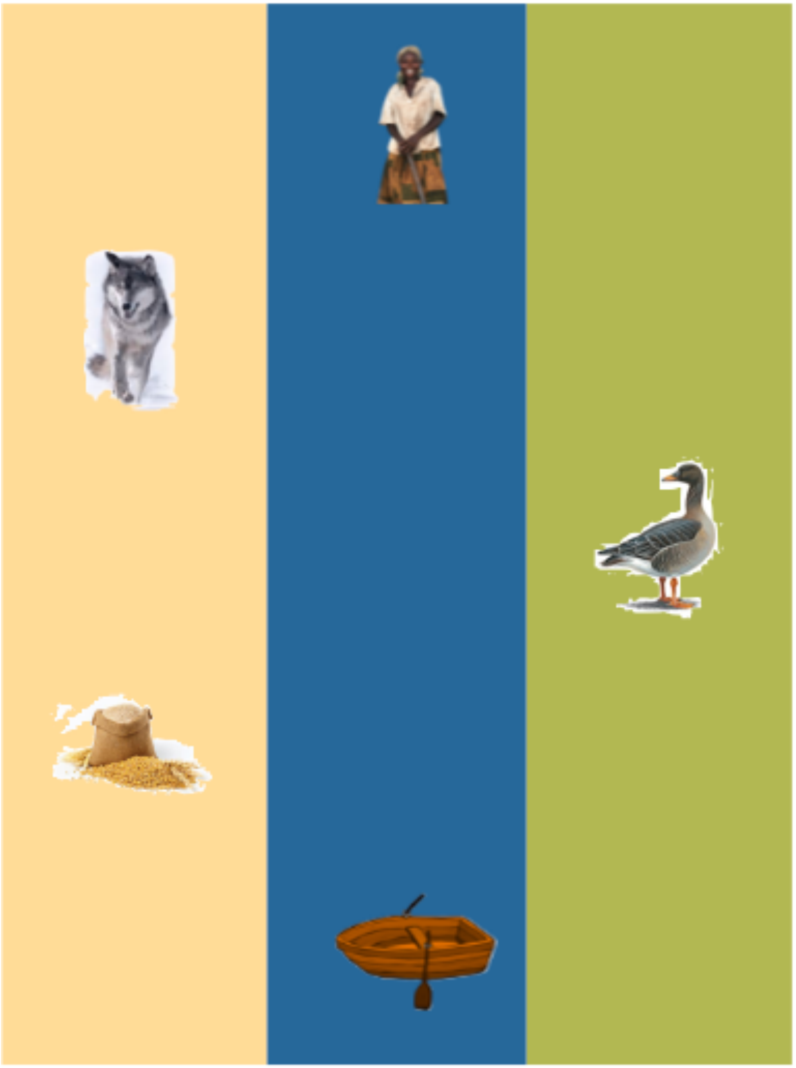

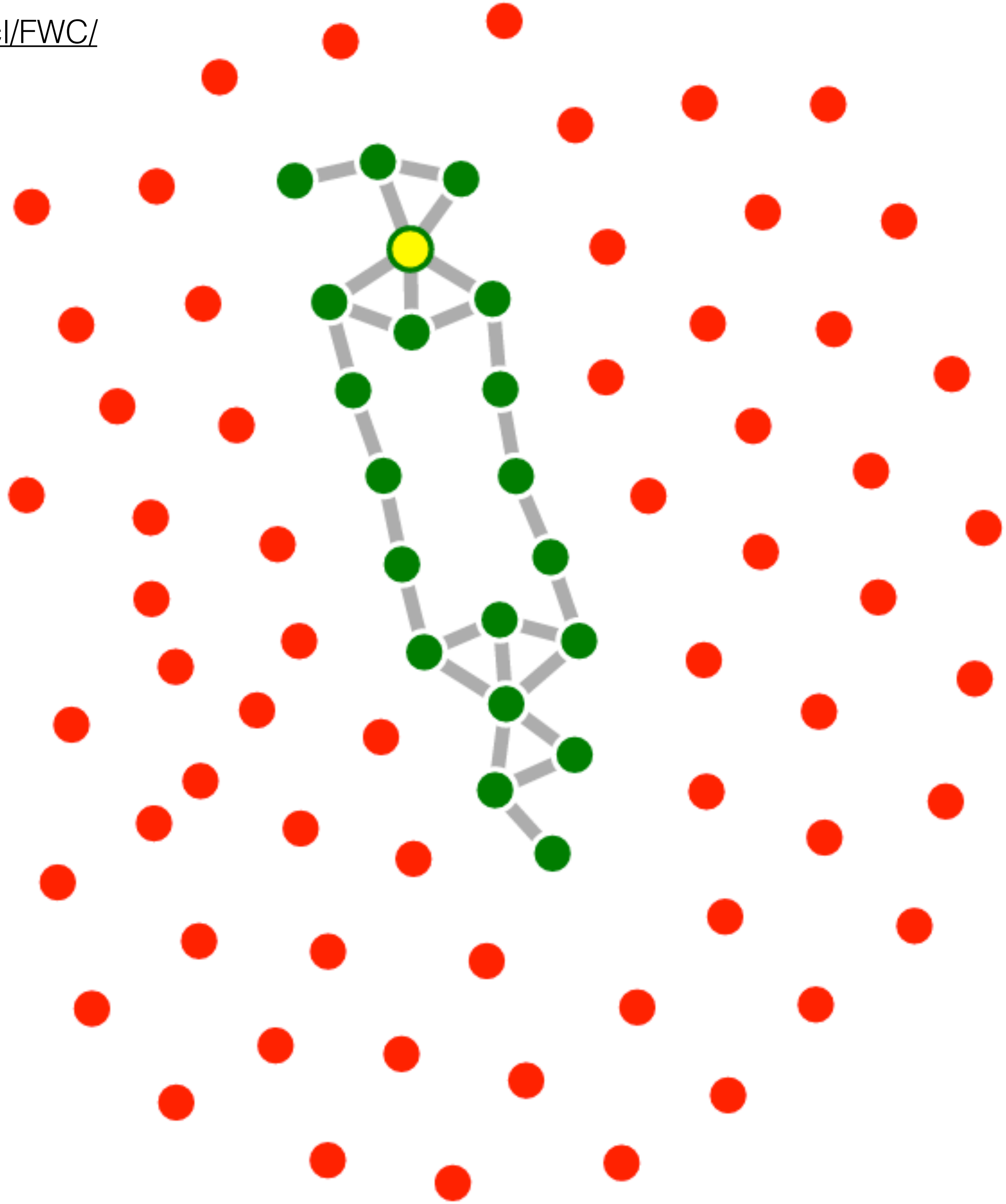

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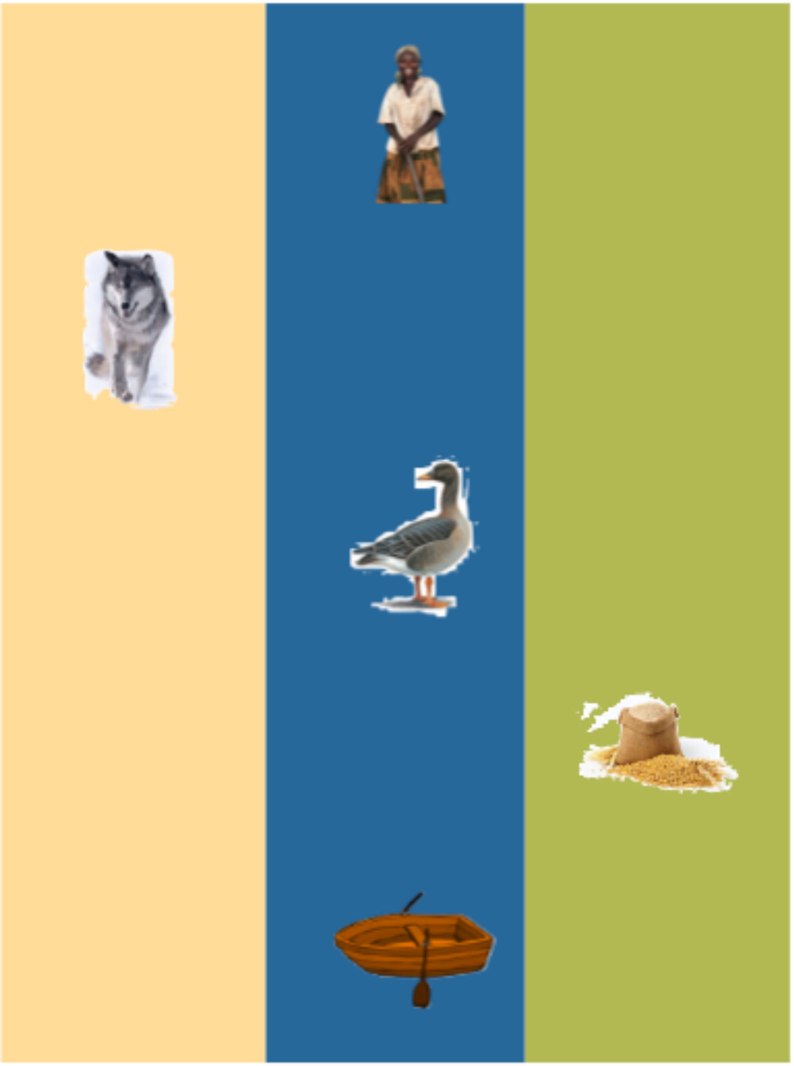

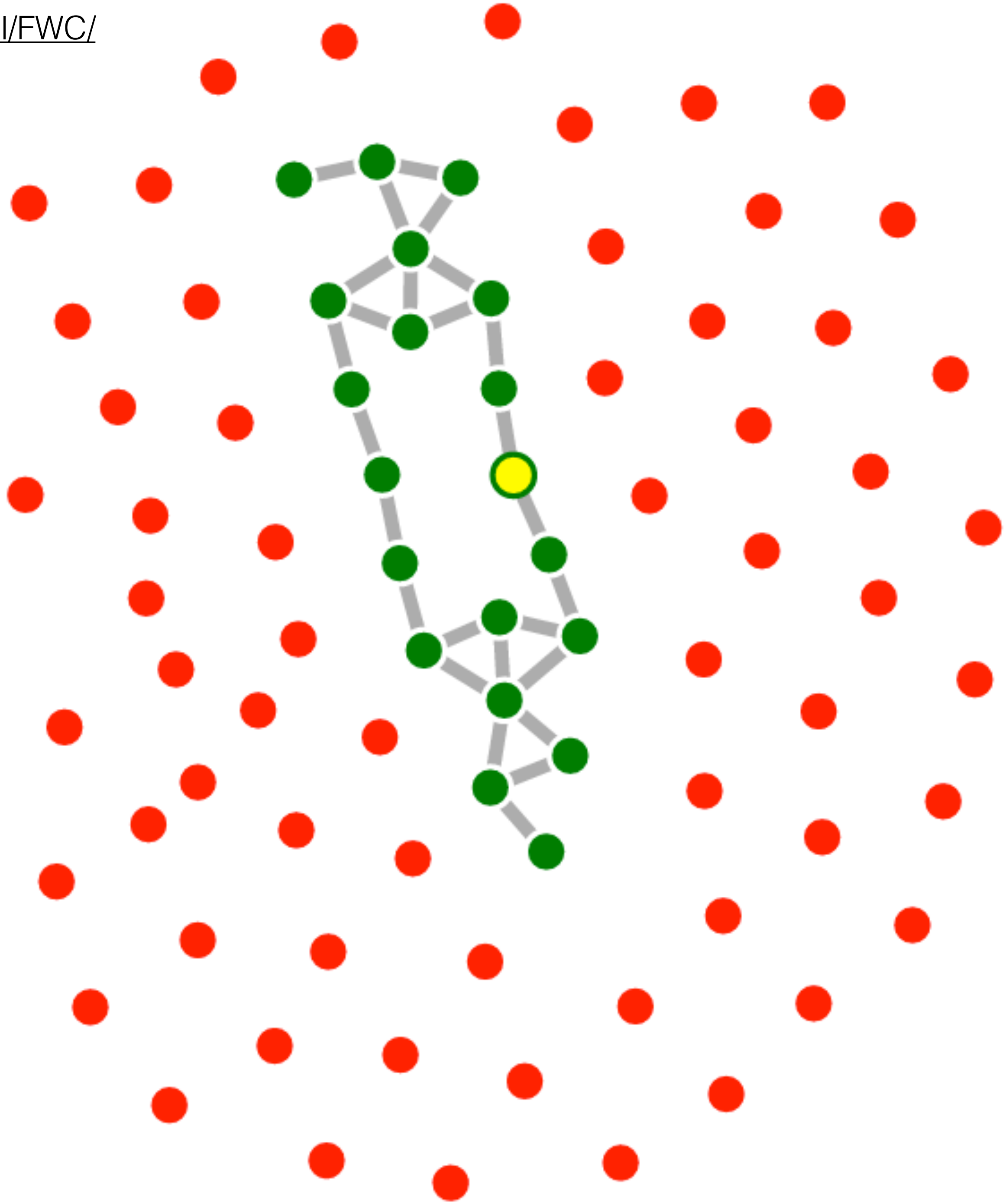

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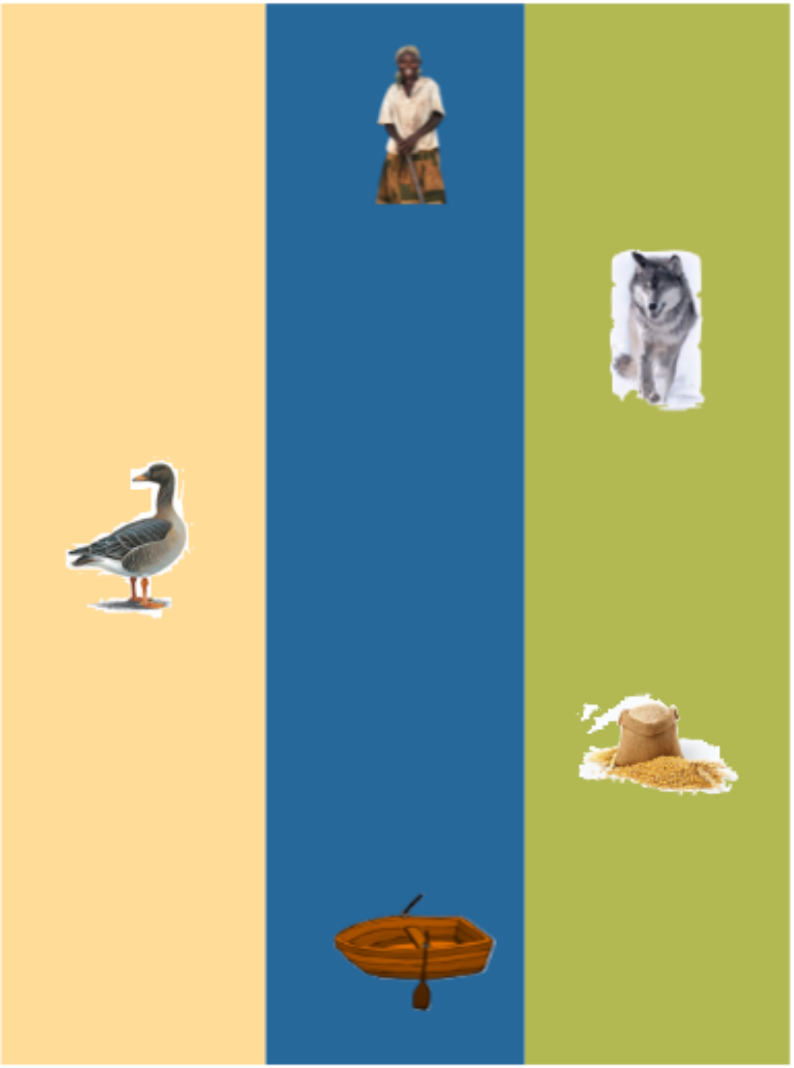

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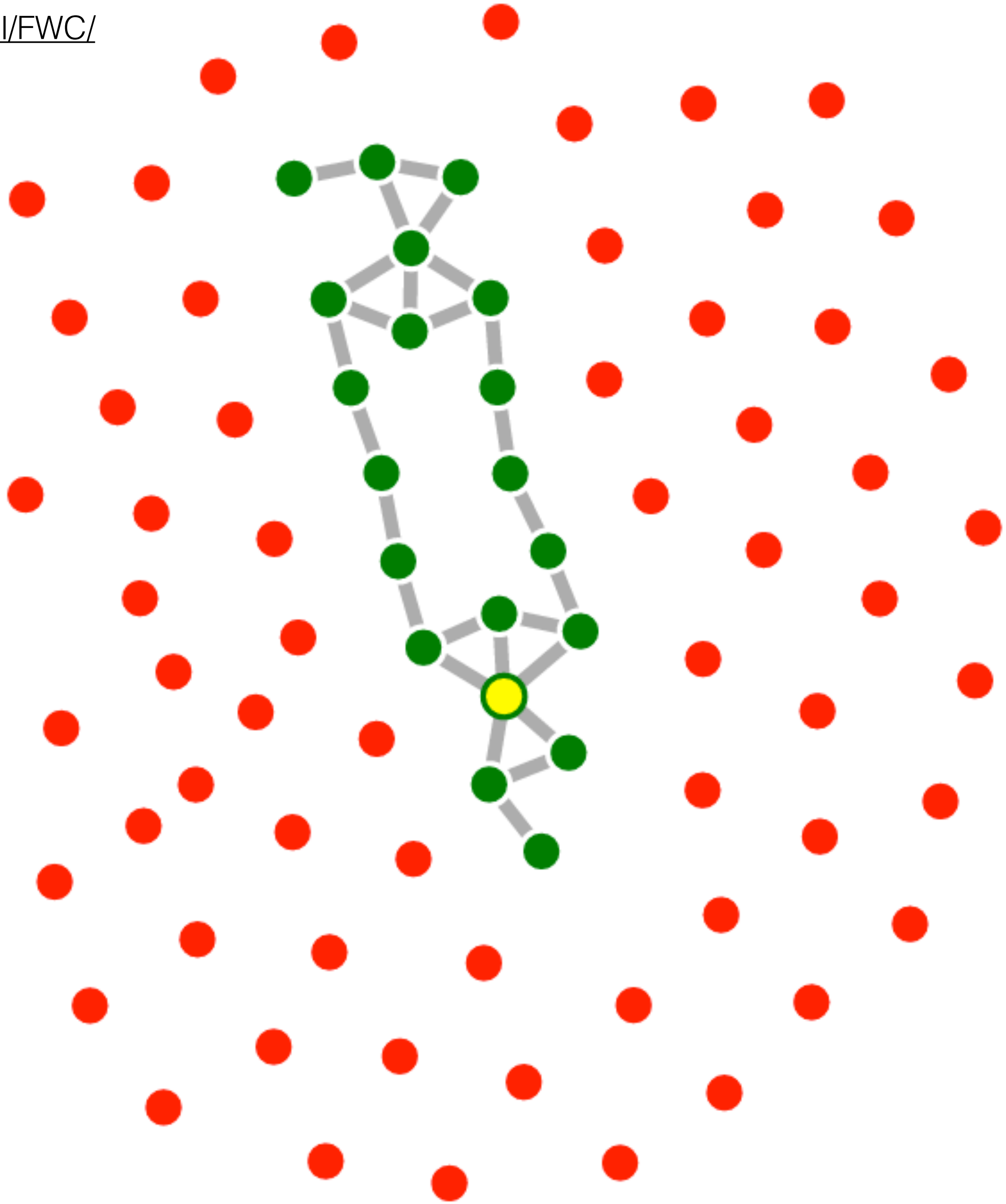

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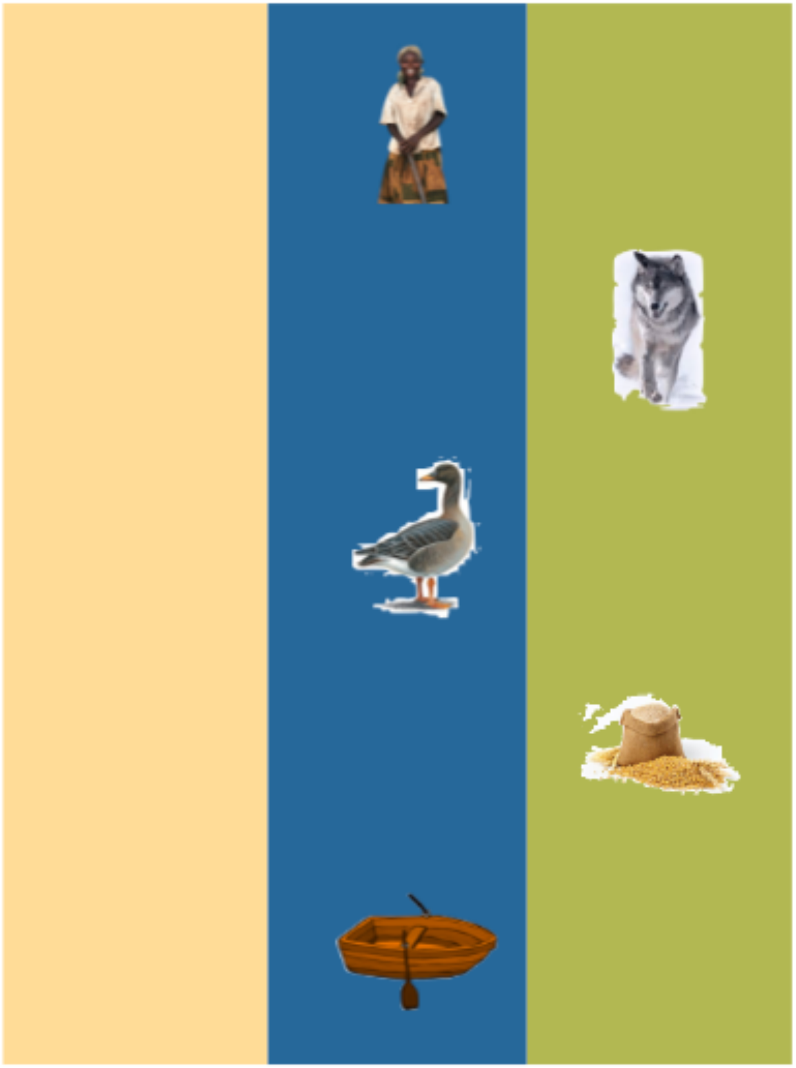

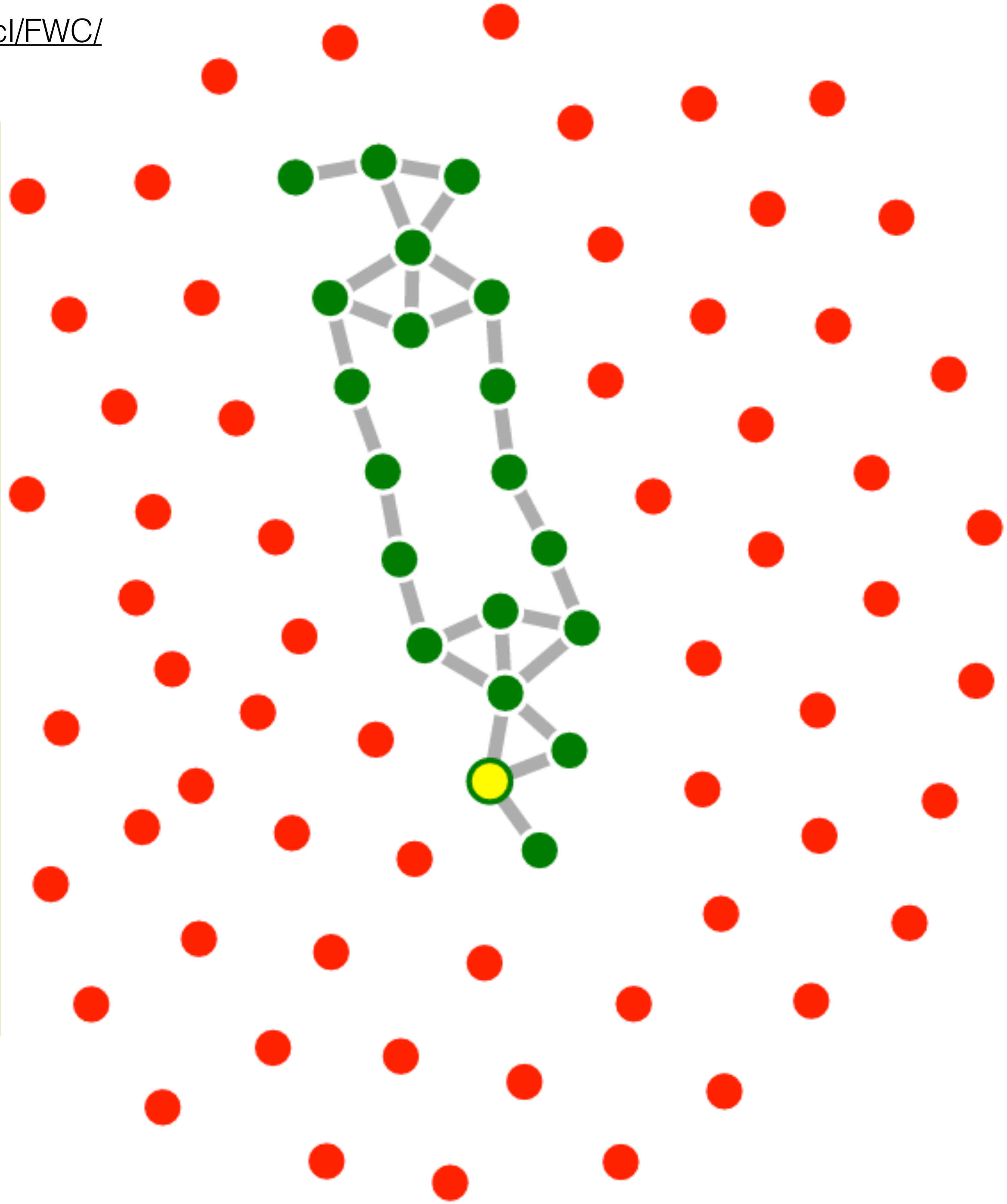

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How can we use logic to specify the transitions?

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

one place	$(WW \oplus WB \oplus WE) \wedge \neg(WW \wedge WB \wedge WE)$
not solo	$GB \rightarrow FB$
no conflict	$GW \wedge (WW \vee CW) \rightarrow FW$
no overload	$\neg(GB \wedge CB) \wedge \neg(GB \wedge WB) \wedge \neg(WB \wedge CB)$

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

one place	$(WW \oplus WB \oplus WE) \wedge \neg(WW \wedge WB \wedge WE)$	$\times 4$ (wolf,goose,corn,farmer)
not solo	$GB \rightarrow FB$	$\times 3$ (wolf,goose,corn)
no conflict	$GW \wedge (WW \vee CW) \rightarrow FW$	$\times 2$ (east, west)
no overload	$\neg(GB \wedge CB) \wedge \neg(GB \wedge WB) \wedge \neg(WB \wedge CB)$	$\times 1$

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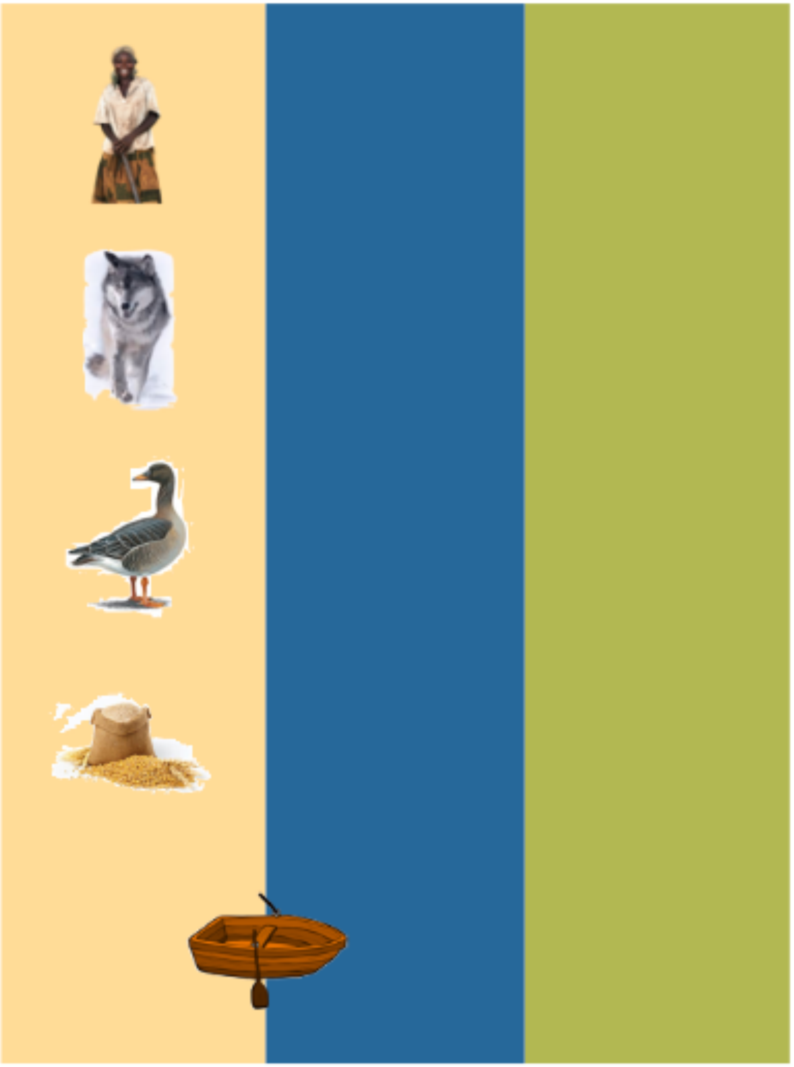

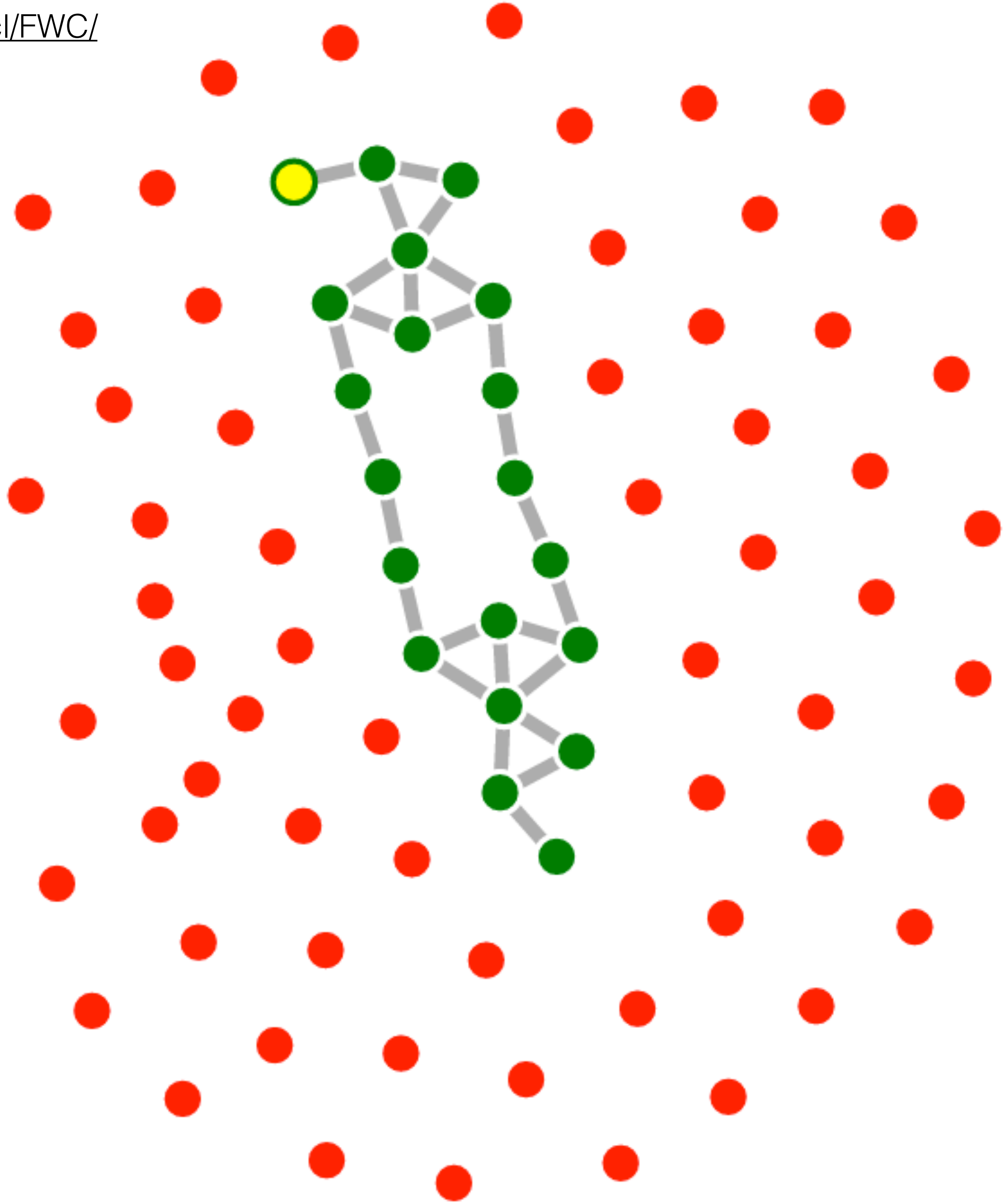

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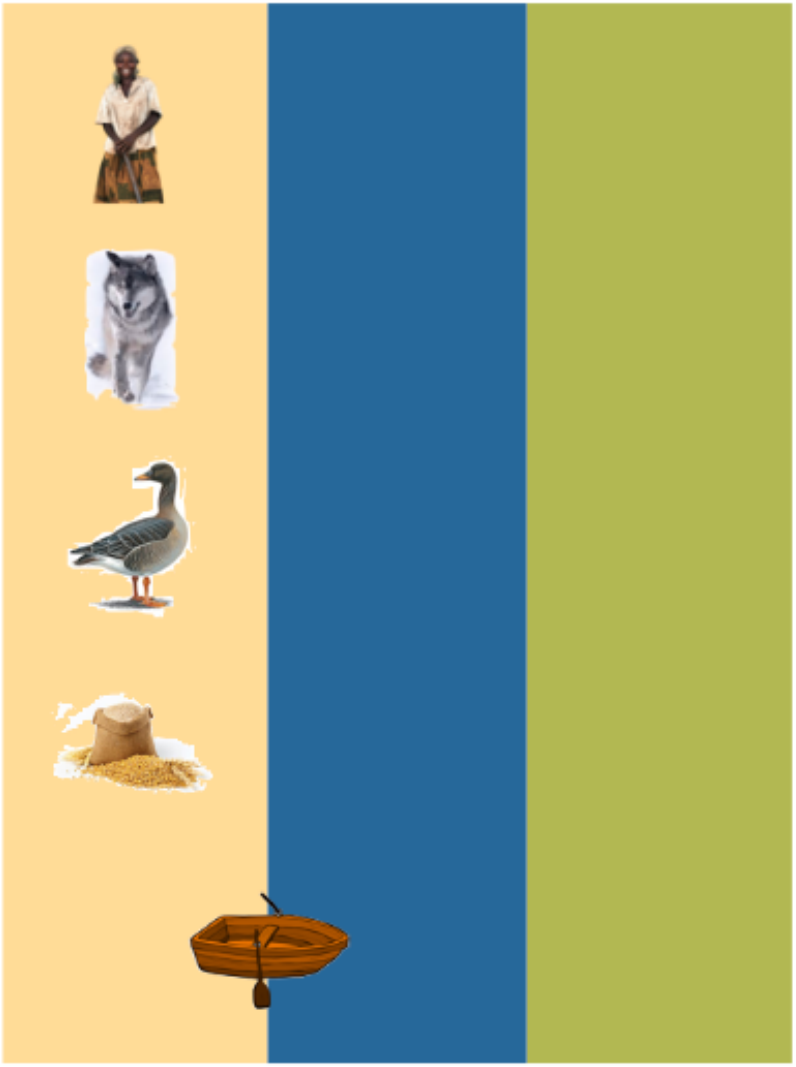

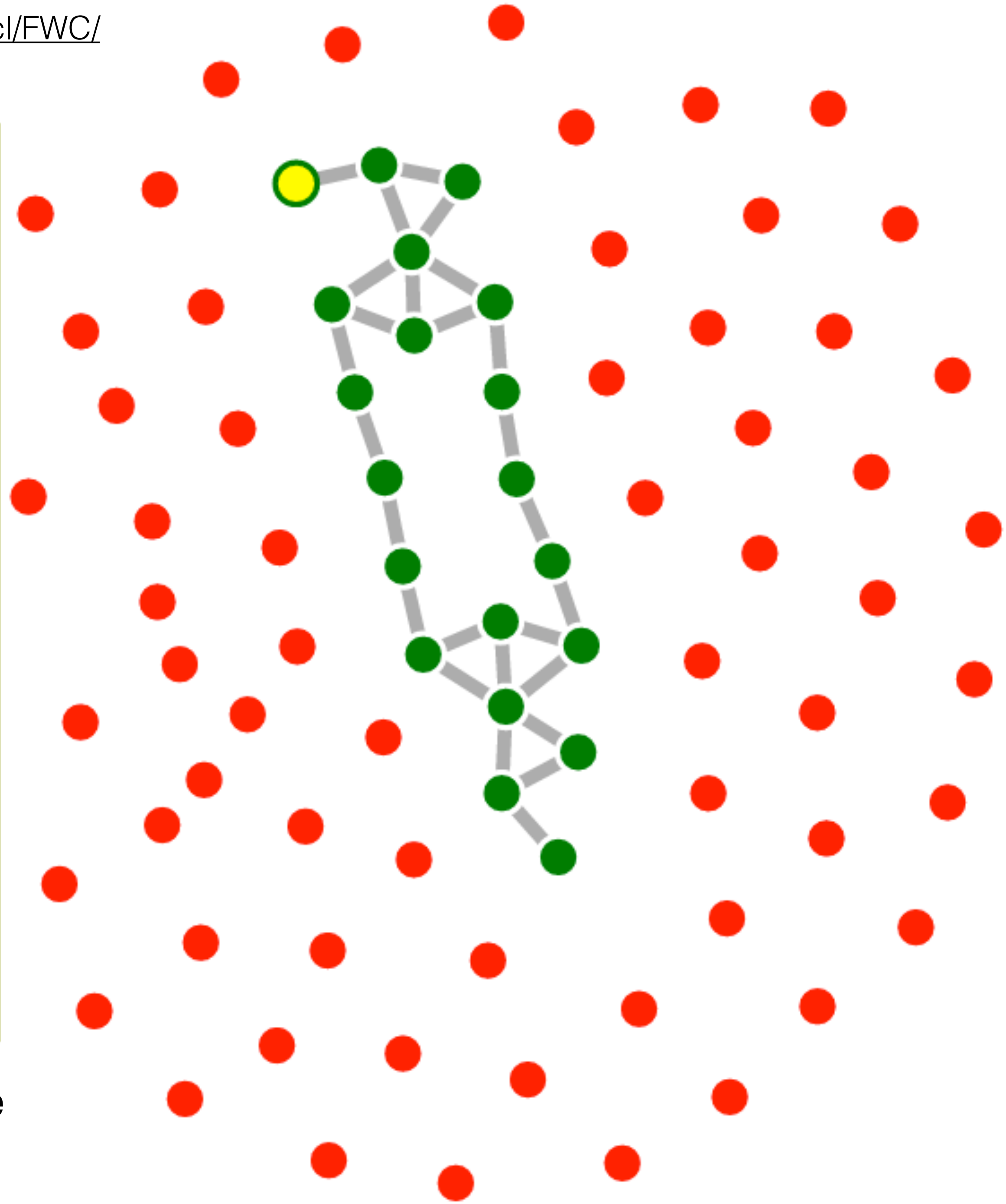

How can we use logic to specify the transitions?

This is a **non-deterministic** system. We define a next state **relation**.

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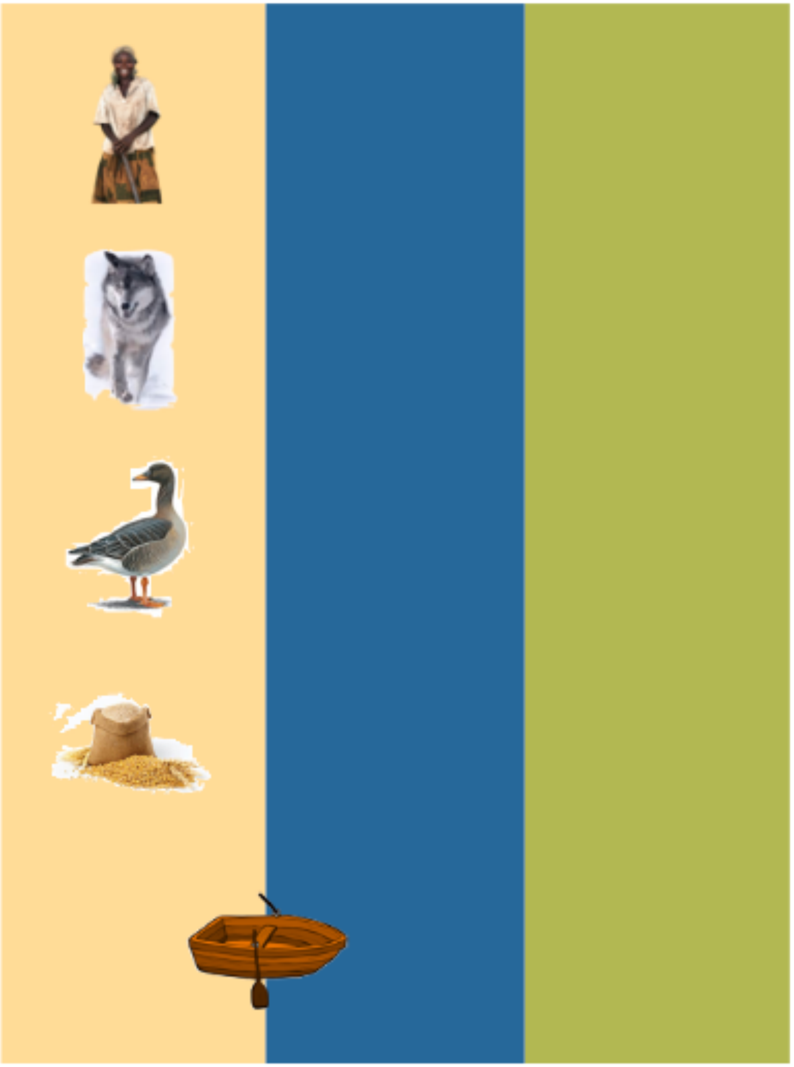

How can we use logic to specify the transitions?

This is a **non-deterministic** system.
We define a next state **relation**.

Again we introduce next state variables WW' etc.

Here we have
$FW \wedge WW \wedge GW \wedge CW$

Is it possible that WE' ?

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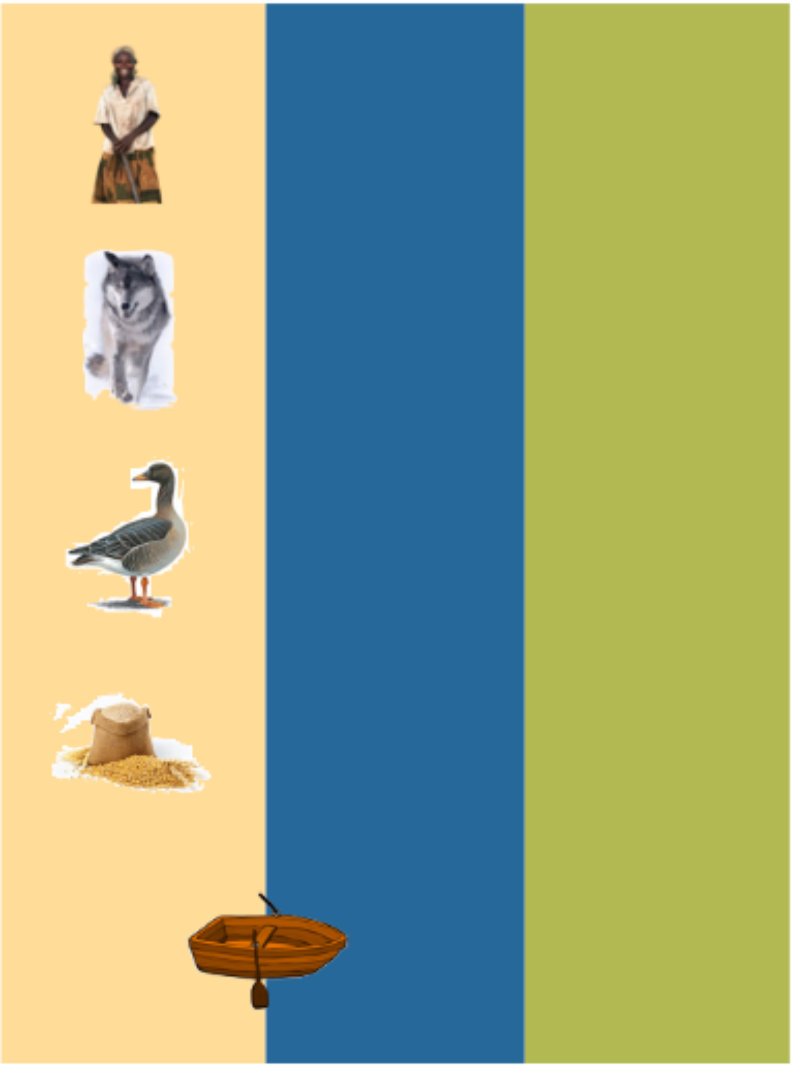

How can we use logic to specify the transitions?

This is a **non-deterministic** system. We define a next state **relation**.

We introduce next state variables WW' etc. and give conditions on the next state.

Here we have
$FW \wedge WW \wedge GW \wedge CW$

Is it possible that WE' ? **NO**

One thing true in our model is that
$WE' \rightarrow WE \vee WB$

What else do we need to say to give a complete description ?

What does it mean for a description to be complete?

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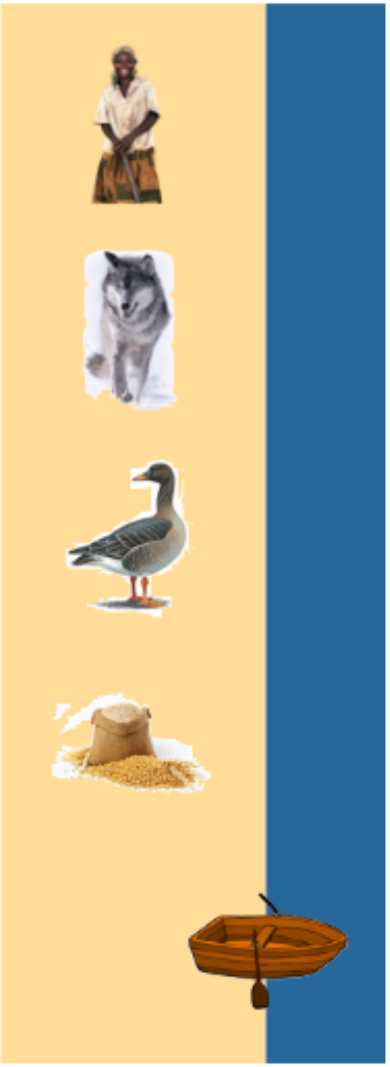

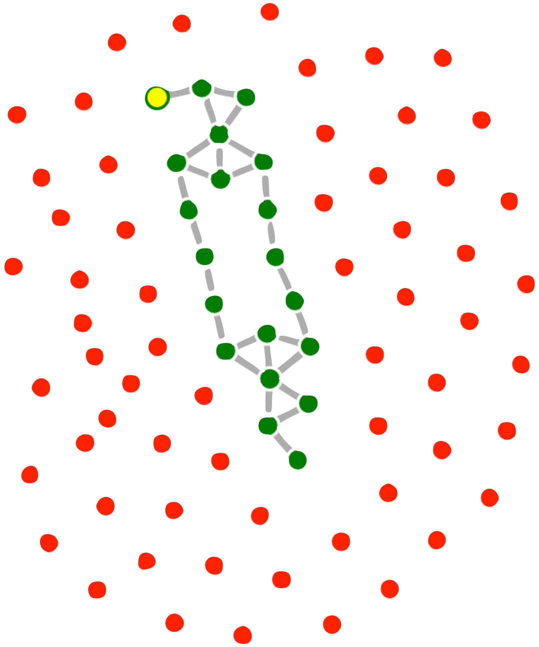

How can we use logic to specify the transitions?

This is a **non-deterministic** system.
We define a next state relation.

We introduce next state variables WW′ etc. and give conditions on the next state.

We require:

$FE' \rightarrow FE \lor FB$	$FW' \rightarrow FW \lor FB$
$WE' \rightarrow WE \lor WB$	$WW' \rightarrow WW \lor WB$
$GE' \rightarrow GE \lor GB$	$GW' \rightarrow GW \lor GB$
$CE' \rightarrow CE \lor CB$	$CW' \rightarrow CW \lor CB$

There is a transition between a pair of states iff these conditions are satisfied

What does it mean for a description to be complete?