Informatics 1

Lecture 8 Resolution

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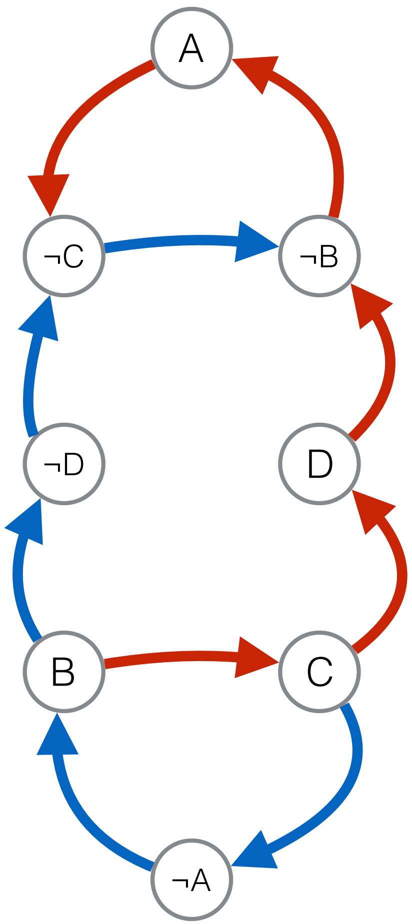
Binary constraints

You may not take both Archeology and Chemistry
If you take Biology you must take Chemistry
You must take Biology or Archeology
If you take Chemistry you must take Divinity
You may not take both Divinity and Biology

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

We represent 5 clauses (constraints)
by 10 arrows to give a directed graph.



Here we have 4 atoms A, B, C, D ; so 8 literals.
Any valuation makes 4 literals true; 4 literals false.

The valuation satisfies the constraints
provided no arrow goes from $\top$ to $\bot$.

In this case, we can arrange the diagram with a
line that satisfies the arrow rule,
and separates each atom from its negation.

$$(\neg A \lor \neg C) \land (\neg B \lor C) \land (B \lor A) \land (\neg C \lor D) \land (\neg D \lor \neg B)$$

$$\equiv$$

$$(A \to \neg C) \land (B \to C) \land (\neg B \to A) \land (C \to D) \land (D \to \neg B)$$

$$\equiv$$

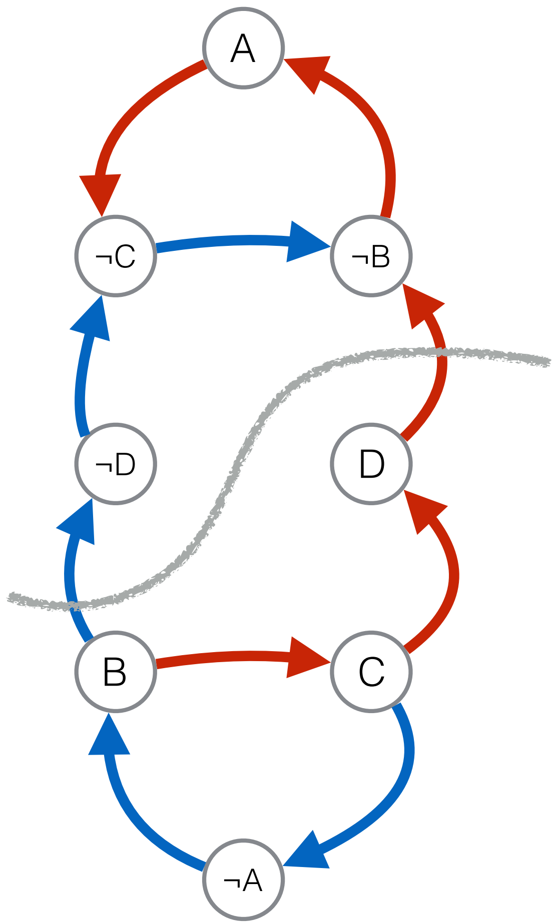
$$(C \to \neg A) \land (\neg C \to \neg B) \land (\neg A \to B) \land (\neg D \to \neg C) \land (B \to \neg D)$$

The valuation satisfies the constraints provided no arrow goes from $\top$ to $\bot$.

In this case, we can arrange the diagram so we can draw a line that satisfies the arrow rule, and separates each atom from its negation.

For this example, there are only two such lines.

The one shown here makes A true and B, C, D all false



$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$\equiv$$

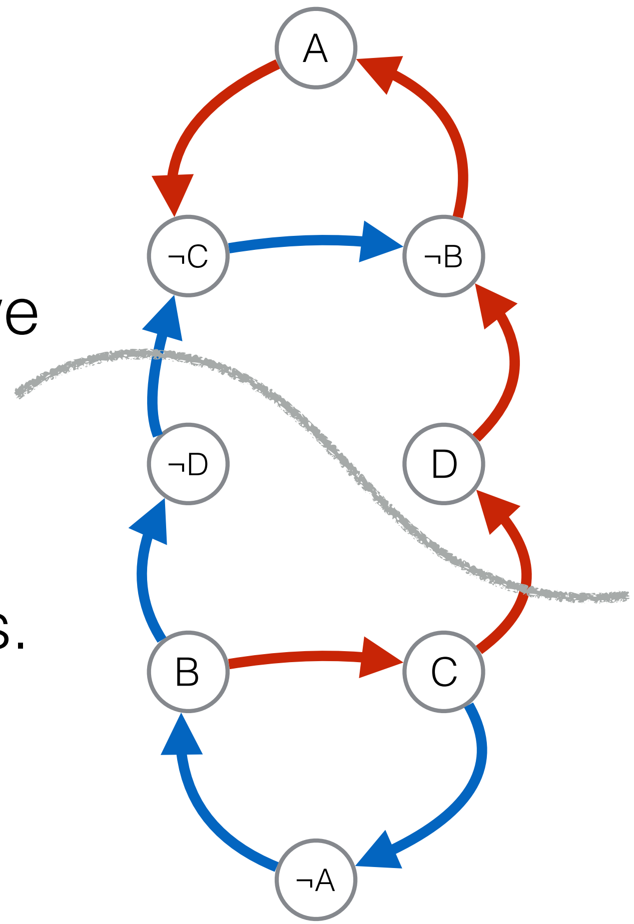
$$(C \rightarrow \neg A) \wedge (\neg C \rightarrow \neg B) \wedge (\neg A \rightarrow B) \wedge (\neg D \rightarrow \neg C) \wedge (B \rightarrow \neg D)$$

The valuation satisfies the constraints provided no arrow goes from ⊤ to ⊥.

In this case, we can arrange the diagram so we can draw a line that satisfies the arrow rule, and separates each atom from its negation.

For this example, there are only two such lines.

The one shown here makes A, D true and B, C false



$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

$$\equiv$$

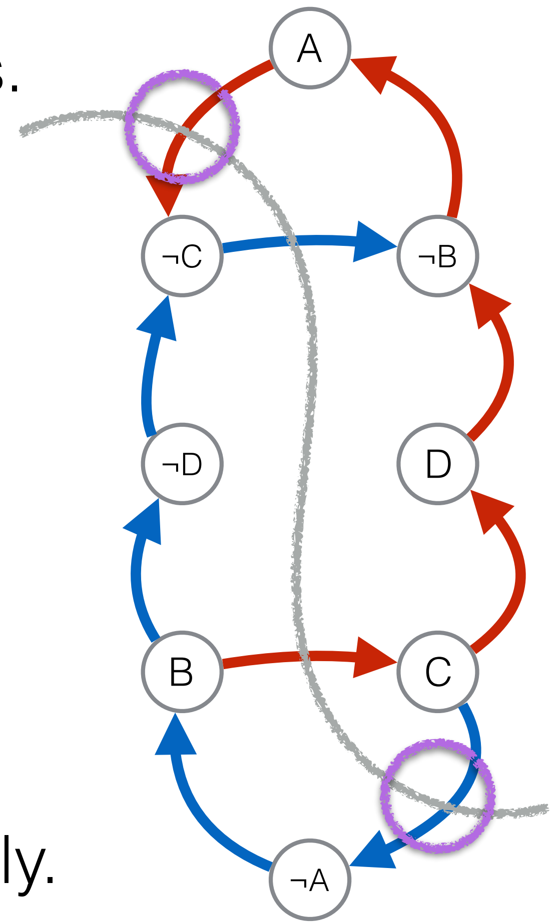
$$(C \rightarrow \neg A) \wedge (\neg C \rightarrow \neg B) \wedge (\neg A \rightarrow B) \wedge (\neg D \rightarrow \neg C) \wedge (B \rightarrow \neg D)$$

For this example, there are only two such lines.

For example, the valuation that makes
A, C, D true and B false
violates the arrow rule

Our analysis shows that our constraints are
equivalent to the requirement
$A \wedge \neg B \wedge \neg C$

We will find a way to compute this result logically.



$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$\equiv$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

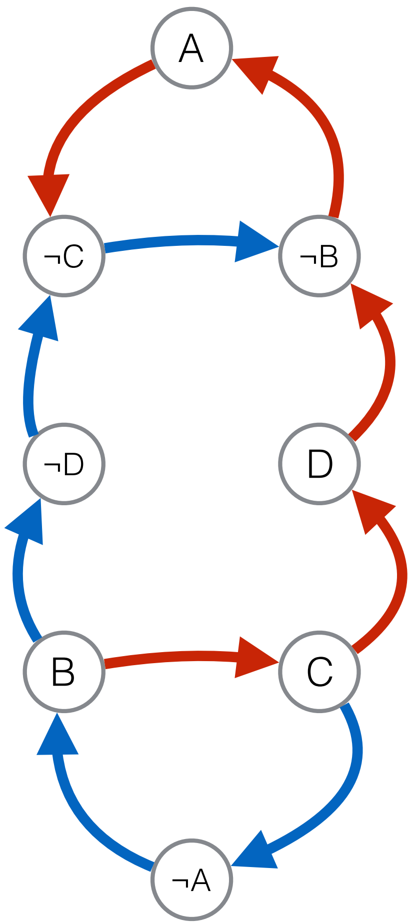
$$\equiv$$

$$(C \rightarrow \neg A) \wedge (\neg C \rightarrow \neg B) \wedge (\neg A \rightarrow B) \wedge (\neg D \rightarrow \neg C) \wedge (B \rightarrow \neg D)$$

Our analysis shows that our constraints are equivalent to the requirement

$A \wedge \neg B \wedge \neg C$

Two expressions are equivalent iff they are satisfied by the same valuations.

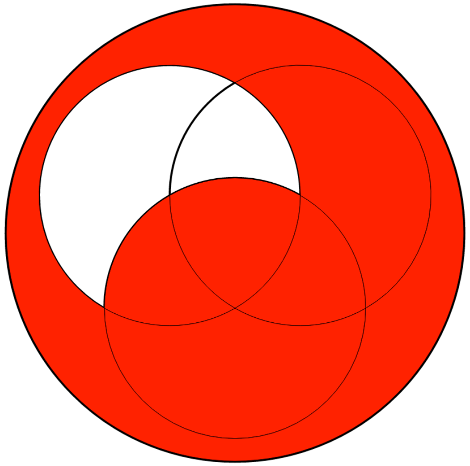


$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

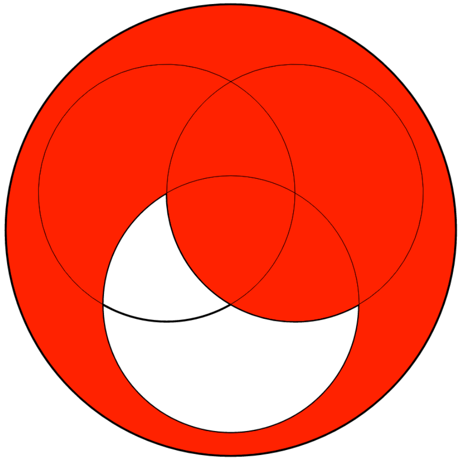
$$\equiv$$

$$A \wedge \neg B \wedge \neg C$$

$R \to G$ $\quad$ $G \to A$ $\quad$ $(R \to G) \wedge (G \to A)$



$\wedge$



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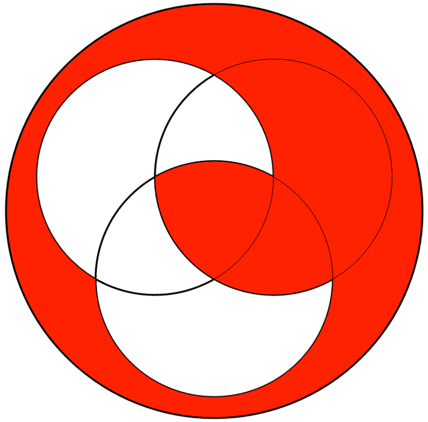
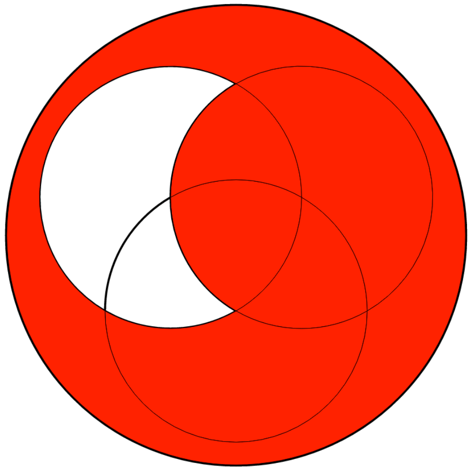


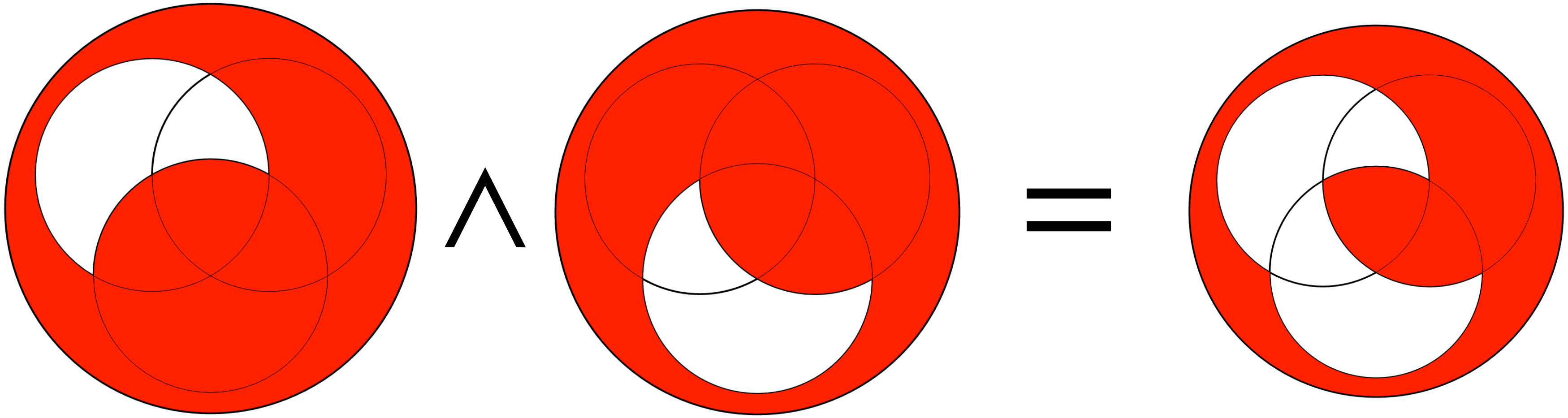
$\Downarrow$

$$\frac{R \to G \qquad G \to A}{R \to A}$$



$R \to A$

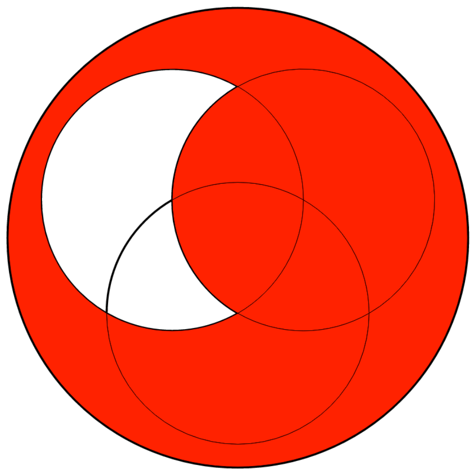
$R \rightarrow G$ $\quad$ $G \rightarrow A$ $\quad$ $(R \rightarrow G) \wedge (G \rightarrow A)$



$\Downarrow$

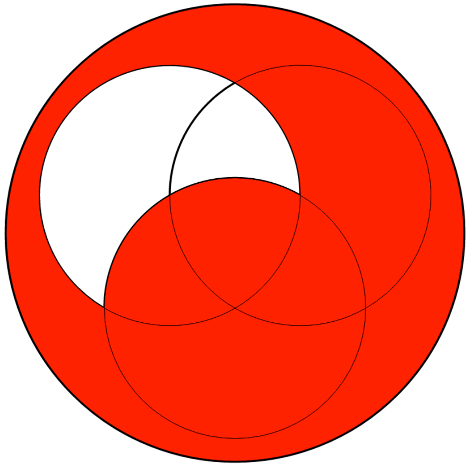
Each condition excludes some valuations

If we exclude valuations where $R \wedge \neg G$
and we exclude valuations where $G \wedge \neg A$
then
we exclude valuations where $R \wedge \neg A$

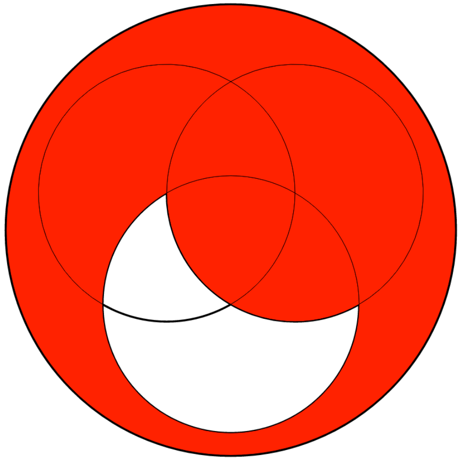


$R \rightarrow A$

$R \to G$ $\qquad$ $G \to A$ $\qquad$ $(R \to G) \land (G \to A)$



$\land$



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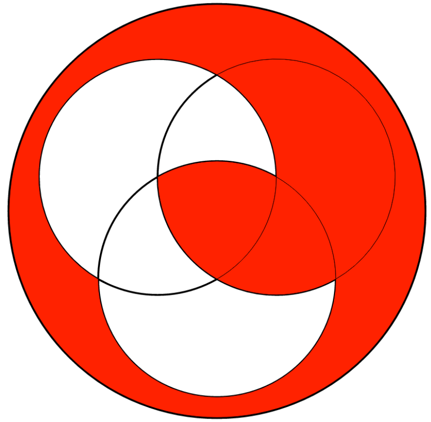
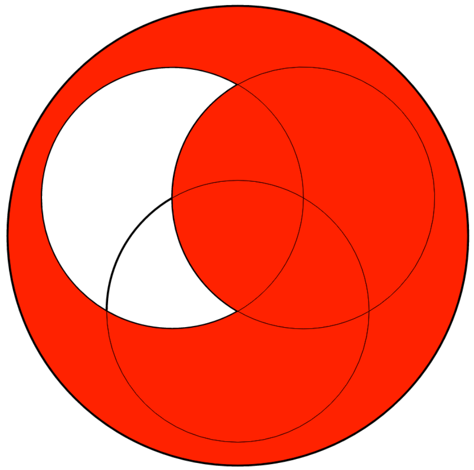


$\Downarrow$

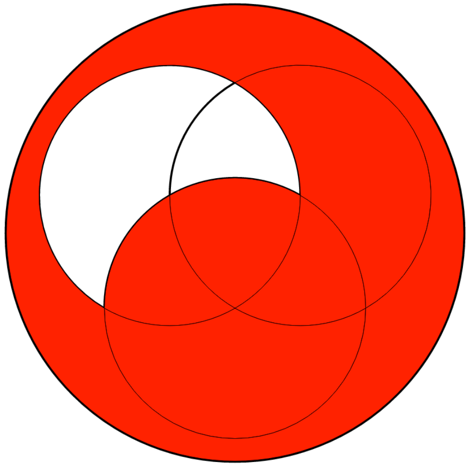


$R \to A$

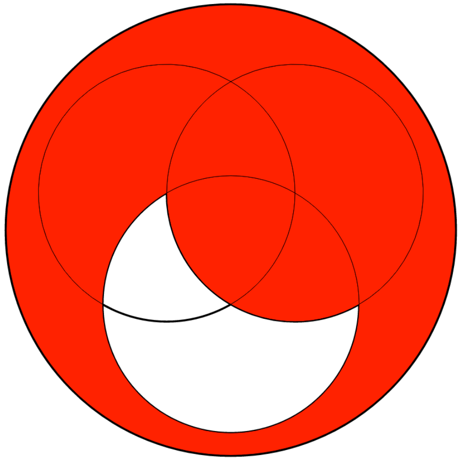
If we exclude valuations where R∧¬G
and we exclude valuations where G∧¬A
then
we exclude valuations where R∧¬A

$$\frac{\neg(R \land \neg G) \qquad \neg(G \land \neg A)}{\neg(R \land \neg A)}$$

$R \to G$ $\qquad$ $G \to A$ $\qquad$ $(R \to G) \wedge (G \to A)$



$\wedge$



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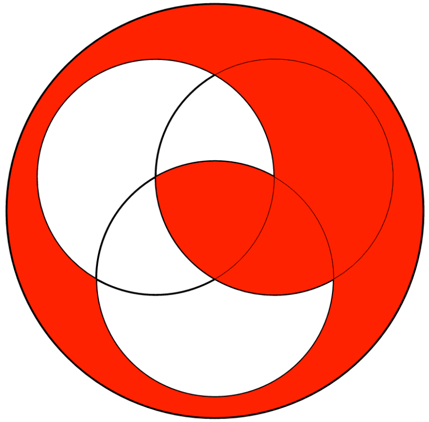
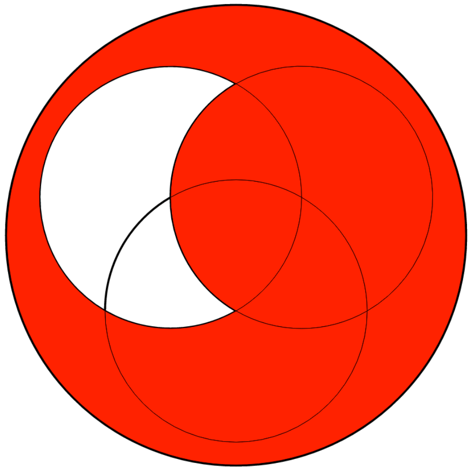


$\Downarrow$

$$\frac{\neg(R \wedge \neg G) \qquad \neg(G \wedge \neg A)}{\neg(R \wedge \neg A)}$$

$$\frac{\neg R \vee G \qquad \neg G \vee A}{\neg R \vee A}$$



$R \to A$

Resolution (prototype)

$$\frac{\neg R \vee G \qquad\qquad \neg G \vee A}{\neg R \vee A}$$

If a valuation satisfies the constraints above the line
then it satisfies the constraint below the line.

We say the rule is **sound**.

Resolution (prototype)

$$\frac{\neg R \lor G \qquad \neg G \lor A}{\neg R \lor A}$$

$$\frac{X \lor G \qquad \neg G \lor Y}{X \lor Y}$$

Resolution (v 0.1)

$$\frac{X \vee G \qquad\qquad \neg G \vee Y}{X \vee Y}$$

This rule is **sound**:
if a valuation satisfies the **premises**
(the constraints above the line)
then it satisfies the **conclusion**
(the constraint below the line)

Resolution (examples)

$$\frac{R \vee S \vee T \vee G \qquad \neg G \vee P \vee Q}{R \vee S \vee T \vee P \vee Q}$$

This rule is **sound**:

if a valuation satisfies the premises
(the constraints above the line)
then it satisfies the conclusion
(the constraint below the line)

Resolution (variations)

$$\frac{X \lor G \qquad \neg G}{X \lor Y}$$

$$\frac{G \qquad \neg G \lor Y}{Y}$$

These rules are **sound**:
if a valuation satisfies the premises
(the constraints above the line)
then it satisfies the conclusion
(the constraint below the line)

Resolution (v 0.1)

$$\frac{X \vee G \qquad \neg G \vee Y}{X \vee Y}$$

This rule is **sound**:

if a valuation falsifies the **conclusion**
then it falsifies one of the **premises**

Resolution (v 0.1)

$$\frac{X \vee G \qquad \neg G \vee Y}{X \vee Y}$$

if a valuation, **V**, falsifies the **conclusion**

i.e. if $\mathbf{V}(X \vee Y) = \bot$ then

if $\mathbf{V}(G) = \bot$ then $\mathbf{V}(X \vee G) = \bot$

if $\mathbf{V}(G) = \top$ then $\mathbf{V}(\neg G \vee Y) = \bot$

Resolution (v 1.0)

$$\frac{X \vee G \qquad \neg G \vee Y}{X \vee Y}$$

if a valuation, **V**, falsifies the **conclusion**

i.e. if $\mathbf{V}(X \vee Y) = \bot$ then

if $\mathbf{V}(G) = \bot$ then $\mathbf{V}(X \vee G) = \bot$

if $\mathbf{V}(G) = \top$ then $\mathbf{V}(\neg G \vee Y) = \bot$

Clausal form is a set of sets of literals

$$\{ X_0, X_1, \ldots, X_{n-1} \}$$

where each clause, $X_i = \{ L_0, \ldots, L_{m_i-1} \}$ is a set of literals

Resolution (v. 1.0)

rule for clauses

$$\frac{\mathbf{X} \qquad\qquad \mathbf{Y}}{(\mathbf{X} \cup \mathbf{Y}) \setminus \{\neg A, A\}} \quad \text{where } \neg A \in \mathbf{X},\ A \in \mathbf{Y}$$

If a valuation **V** makes everything in the conclusion false then that valuation must make everything in one or other of the premises false.

If **V**(A) = **true**, then **V** makes everything in **X false**
If **V**(A) = **false**, then **V** makes everything in **Y false**

Resolution (v. 1.0)

rule for clauses

set union

set difference

side-condition

$$\frac{\mathbf{X} \qquad \mathbf{Y}}{(\mathbf{X} \cup \mathbf{Y}) \setminus \{ \neg A, A \}} \quad \text{where } \neg A \in \mathbf{X},\ A \in \mathbf{Y}$$

If a valuation **V** makes everything in the conclusion false then that valuation must make everything in one or other of the premises false.

If **V**(A) = **true**, then **V** makes everything in **X false**
If **V**(A) = **false**, then **V** makes everything in **Y false**

Resolution (v. 1.0)

rule for clauses

if

we start from some set of constraints and apply resolution,

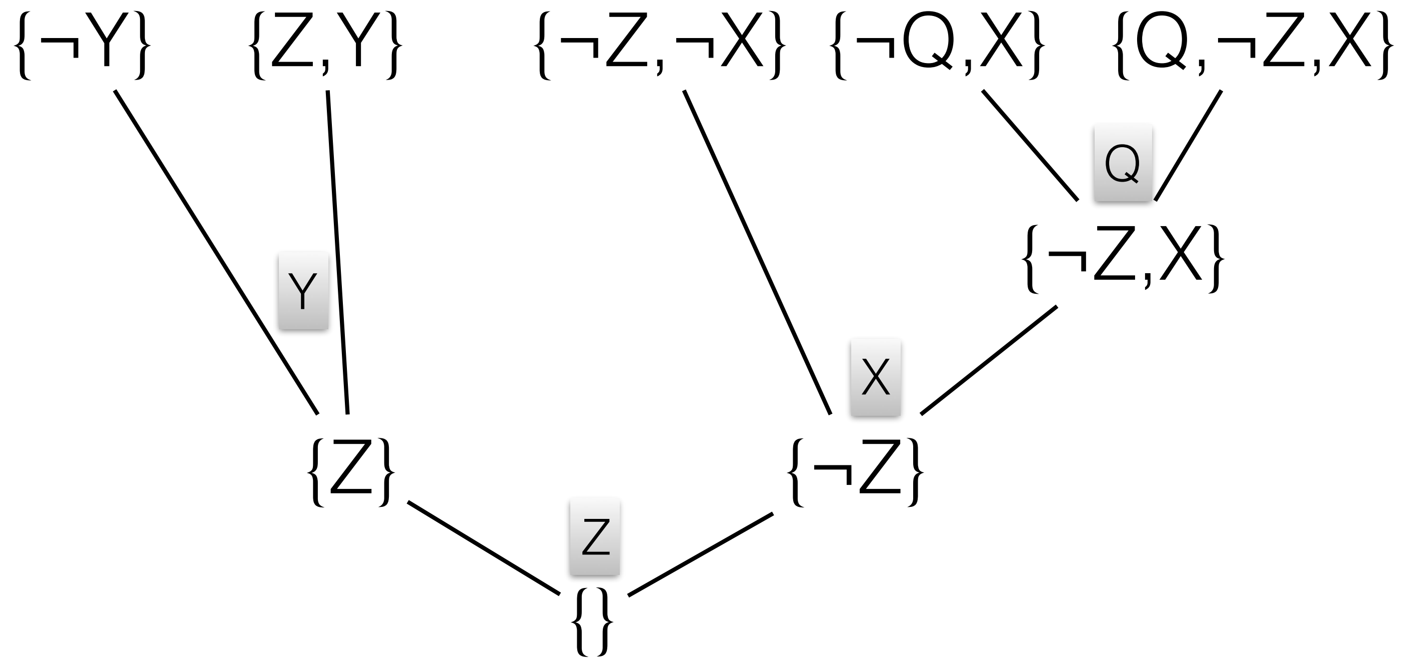
then

adding the new constraints
doesn't exclude any more states

if **V** contradicts any new constraint
then it contradicts some original constraint

Resolution

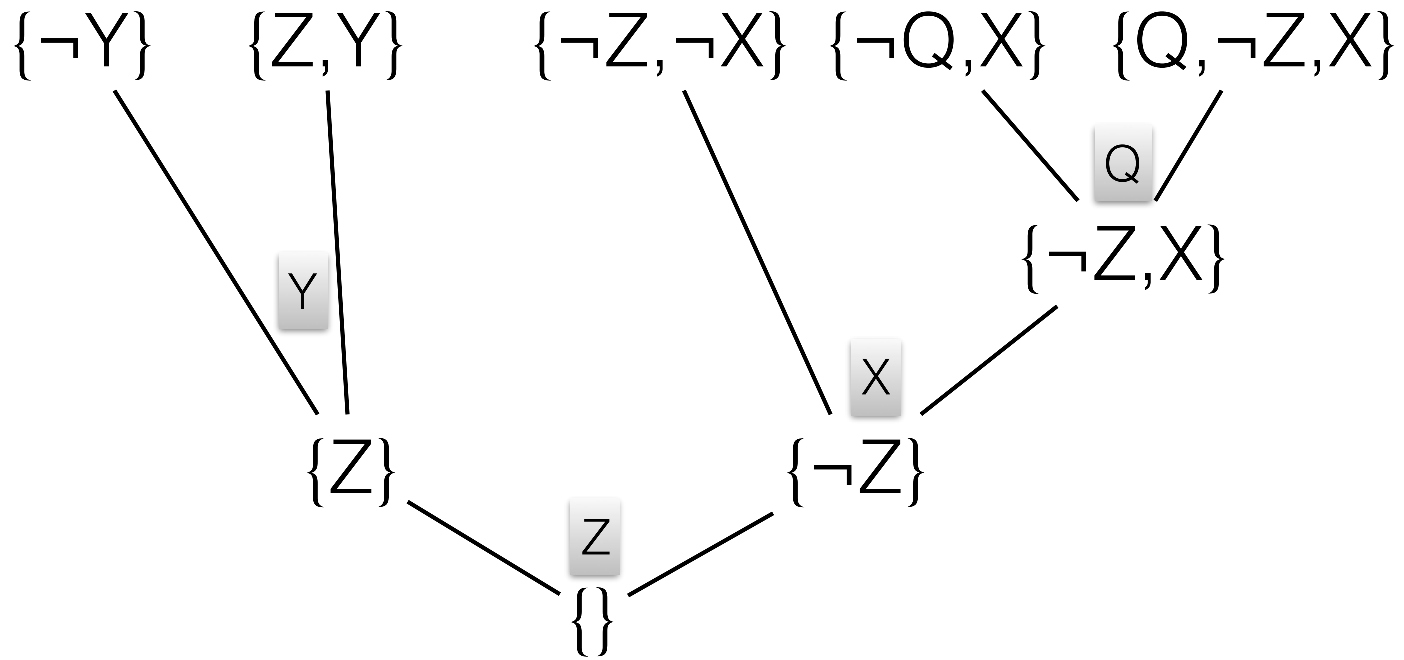
$\mathcal{F}$ = {{¬Y}, {Z,Y}, {¬Z,¬X}, {¬Q,X}, {Q,¬Z,X}}



[**{¬Y}, {Z,Y}, {¬Z,¬X}, {¬Q,X}, {Q,¬Z,X}**,{¬Z,X}, {¬Z}, {Z},{}]

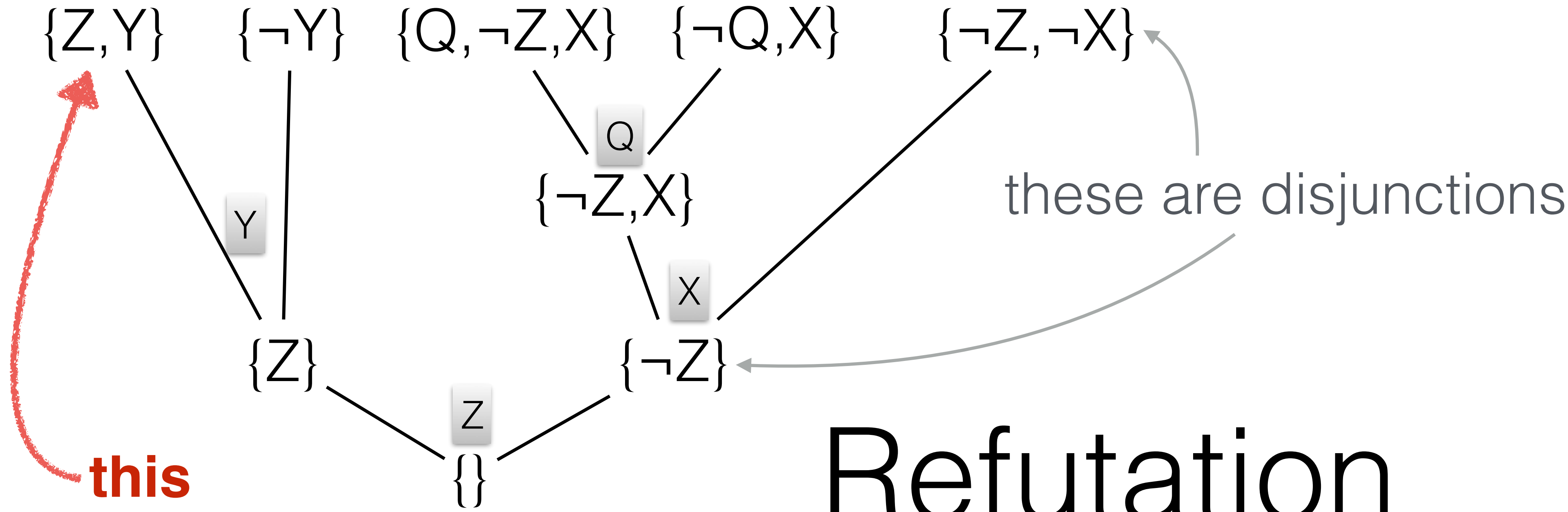
Resolution

F = {{¬Y}, {Z,Y}, {¬Z,¬X}, {¬Q,X}, {Q,¬Z,X}}

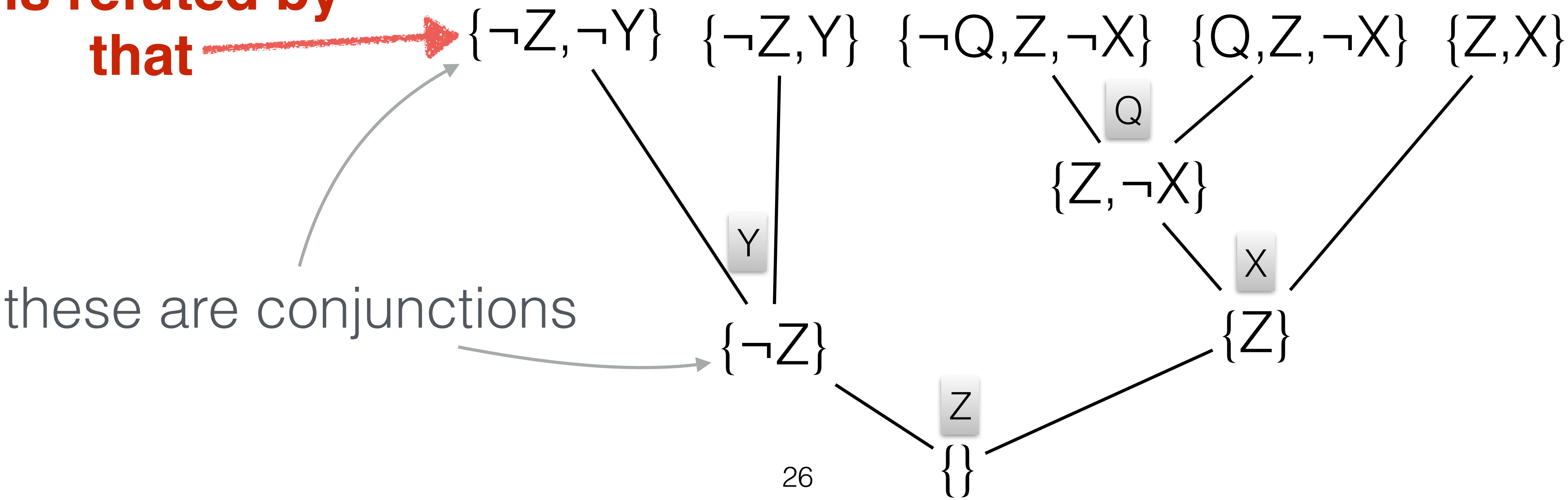


[**{¬Y}, {Z,Y}, {¬Z,¬X}, {¬Q,X}, {Q,¬Z,X}**,{¬Z,X}, {¬Z}, {Z},{}]

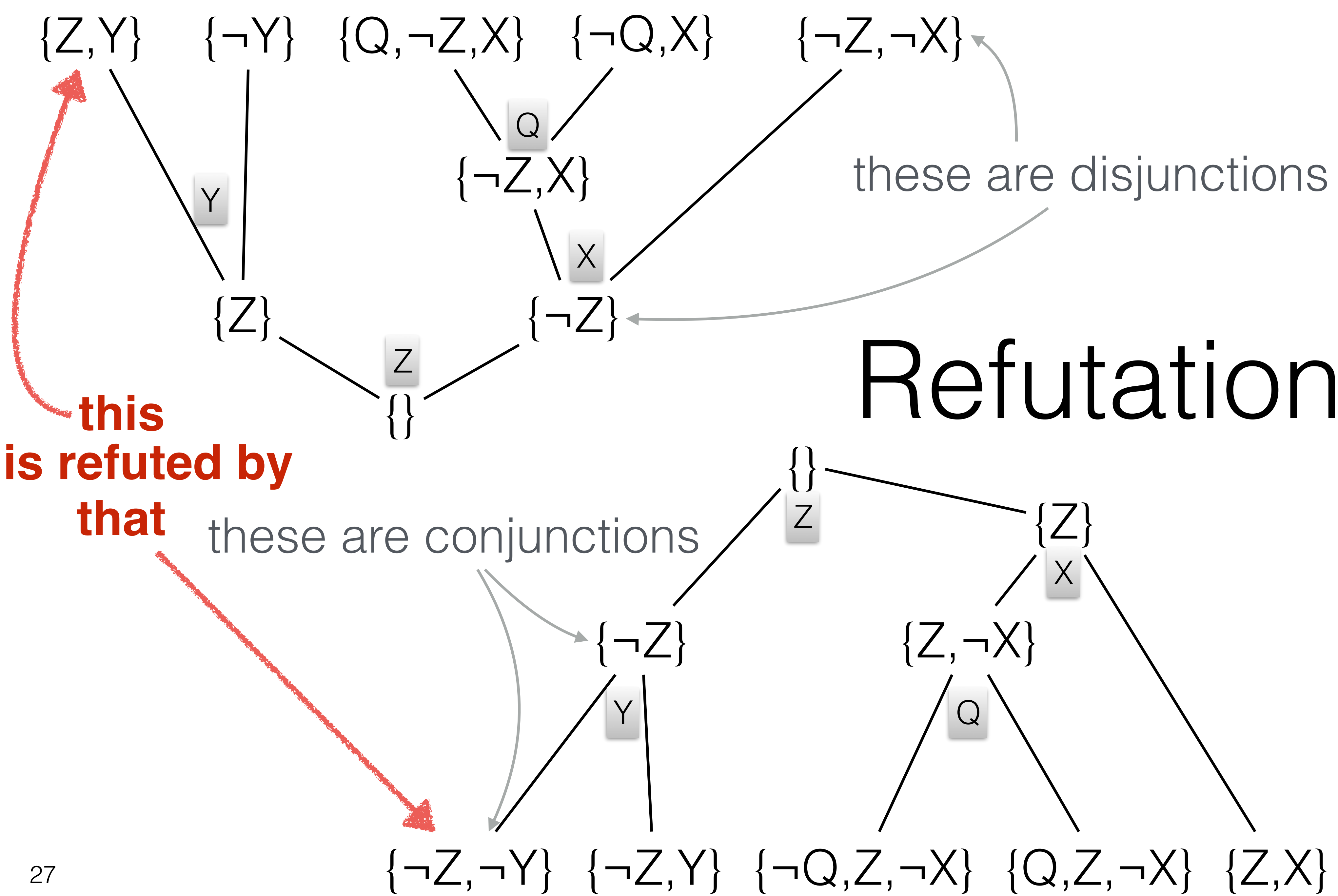
Resolution



Refutation



Resolution



{Z,Y}
{¬Y}
{Q,¬Z,X}
{¬Q,X}
{¬Z,¬X}
Q
{¬Z,X}
Y
X
{Z}
{¬Z}
Z
{}
these are disjunctions
Refutation
this
is refuted by
that
these are conjunctions
{}
Z
{Z}
X
{¬Z}
{Z,¬X}
Y
Q
{¬Z,¬Y}
{¬Z,Y}
{¬Q,Z,¬X}
{Q,Z,¬X}
{Z,X}

When does resolution stop?
What does a set of clauses look like when there are no opportunities for resolution?

Tomorrow we will see that if resolution stops, without producing the empty clause, then we can construct a satisfying valuation.

This shows that the resolution procedure is **complete** – if a set of constraints is inconsistent we will produce the empty clause and a refutation tree.
Otherwise we can produce a satisfying valuation.