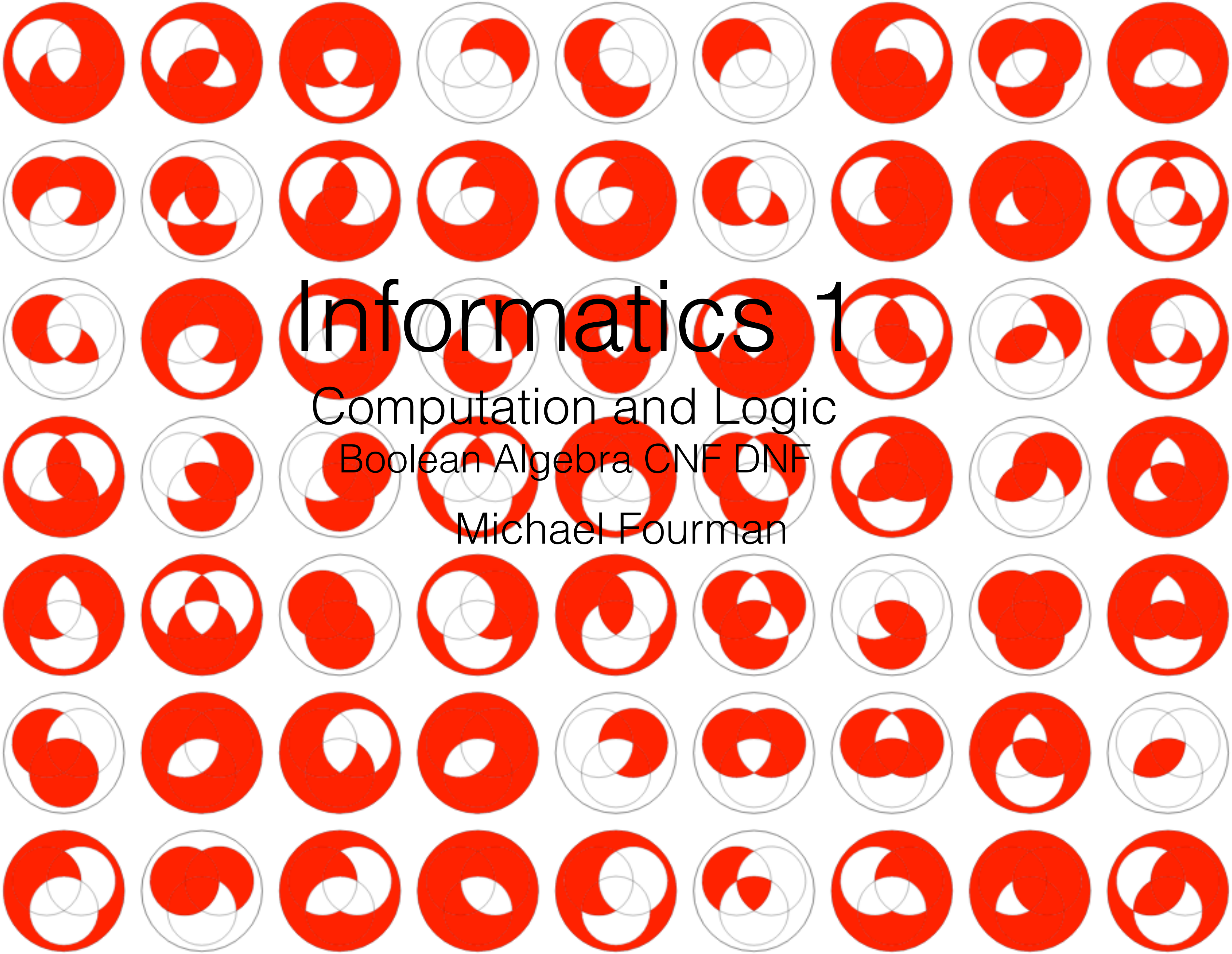

Informatics 1

Computation and Logic

Boolean Algebra CNF DNF

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Basic Boolean operations

$\mathbf{1}, \top$	true, top
$\vee$	disjunction, or
$\wedge$	conjunction, and
$\neg$	negation, not
$\mathbf{0}, \bot$	false, bottom

Boole (1815 – 1864)

$\mathbb{Z}_2 = \{0, 1\}$

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

–	
0	0
1	1

$$x \wedge y \equiv xy$$

$$x \vee y \equiv x + y - xy$$

$$\neg x \equiv 1 - x$$

Here, we use arithmetic mod 2

The same equations work if we use ordinary arithmetic!

∨	0	1
0	0	1
1	1	1

∧	0	1
0	0	0
1	0	1

¬	
0	1
1	0

The algebra of sets

$$\mathcal{P}(S) = \{X \mid X \subseteq S\}$$

$X \vee Y = X \cup Y$	union
$X \wedge Y = X \cap Y$	intersection
$\neg X = S \setminus Y$	complement
$0 = \emptyset$	empty set
$1 = S$	entire set

Derived Operations

Definitions:

$$x \to y \equiv \neg x \vee y \qquad \text{implication}$$

$$x \leftarrow y \equiv x \vee \neg y$$

$$x \leftrightarrow y \equiv (\neg x \wedge \neg y) \vee (x \wedge y) \qquad \text{equality (iff)}$$

$$x \oplus y \equiv (\neg x \wedge y) \vee (x \wedge \neg y) \qquad \text{inequality (xor)}$$

Some equations:

$$x \leftrightarrow y = (x \to y) \wedge (x \leftarrow y)$$

$$x \oplus y = \neg(x \leftrightarrow y)$$

$$x \oplus y = \neg x \oplus \neg y$$

$$x \leftrightarrow y = \neg(x \oplus y)$$

$$x \leftrightarrow y = \neg x \leftrightarrow \neg y$$

an algebraic proof

$$
\begin{aligned}
(x \leftrightarrow y) \leftrightarrow z &= \neg(x \leftrightarrow y) \leftrightarrow \neg z & (a \leftrightarrow b = \neg a \leftrightarrow \neg b) \\
&= (x \oplus y) \leftrightarrow \neg z & (\neg(a \leftrightarrow b) = a \oplus b) \\
&= (x \oplus y) \oplus z & (a \leftrightarrow \neg b = a \oplus b)
\end{aligned}
$$

Boolean connectives

Some equalities:

$$x \vee y = \neg(\neg x \wedge \neg y) \qquad x \wedge y = \neg(\neg x \vee \neg y)$$

$$\neg x = x \rightarrow 0 \qquad x \vee y = \neg x \rightarrow y$$

We will see that $\wedge$, $\vee$, $\neg$ and $\bot$ are sufficient to define any boolean function. These equations show that $\{\wedge, \neg, \bot\}$, $\{\vee, \neg, \bot\}$, and $\{\rightarrow, \bot\}$ are all sufficient sets.

Boolean Algebra

$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	associative
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	distributive
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	commutative
$x \vee 0 = x$	$x \wedge 1 = x$	identity
$x \vee 1 = 1$	$x \wedge 0 = 0$	annihilation
$x \vee x = x$	$x \wedge x = x$	idempotent
$x \vee \neg x = 1$	$\neg x \wedge x = 0$	complements
$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$	absorbtion
$\neg(x \vee y) = \neg x \wedge \neg y$	$\neg(x \wedge y) = \neg x \vee \neg y$	de Morgan
$\neg\neg x = x$	$x \rightarrow y = \neg x \leftarrow \neg y$	

Exercise 2.1

Which of the following rules are *not* valid for arithmetic?
Which of the rules are *not* valid for arithmetic in Z_2?

$x + (y + z) = (x + y) + z$	$x \times (y \times z) = (x \times y) \times z$	associative
$x + (y \times z) = (x + y) \times (x + z)$	$x \times (y + z) = (x \times y) + (x \times z)$	distributive
$x + y = y + x$	$x \times y = y \times x$	commutative
$x + 0 = x$	$x \times 1 = x$	identity
$x + 1 = 1$	$x \times 0 = x$	annihilation
$x + x = x$	$x \times x = x$	idempotent
$x + (x \times y) = x$	$x + (x \times y) = x$	absorbtion
$x + -x = 1$	$x \times -x = 0$	complements

The algebra of sets

$$\mathcal{P}(S) = \{X \mid X \subseteq S\}$$

$X \vee Y = X \cup Y$	union
$X \wedge Y = X \cap Y$	intersection
$\neg X = S \setminus Y$	complement
$0 = \emptyset$	empty set
$1 = S$	entire set

Exercise 2.4 (for mathematicians)

In any Boolean algebra, define,

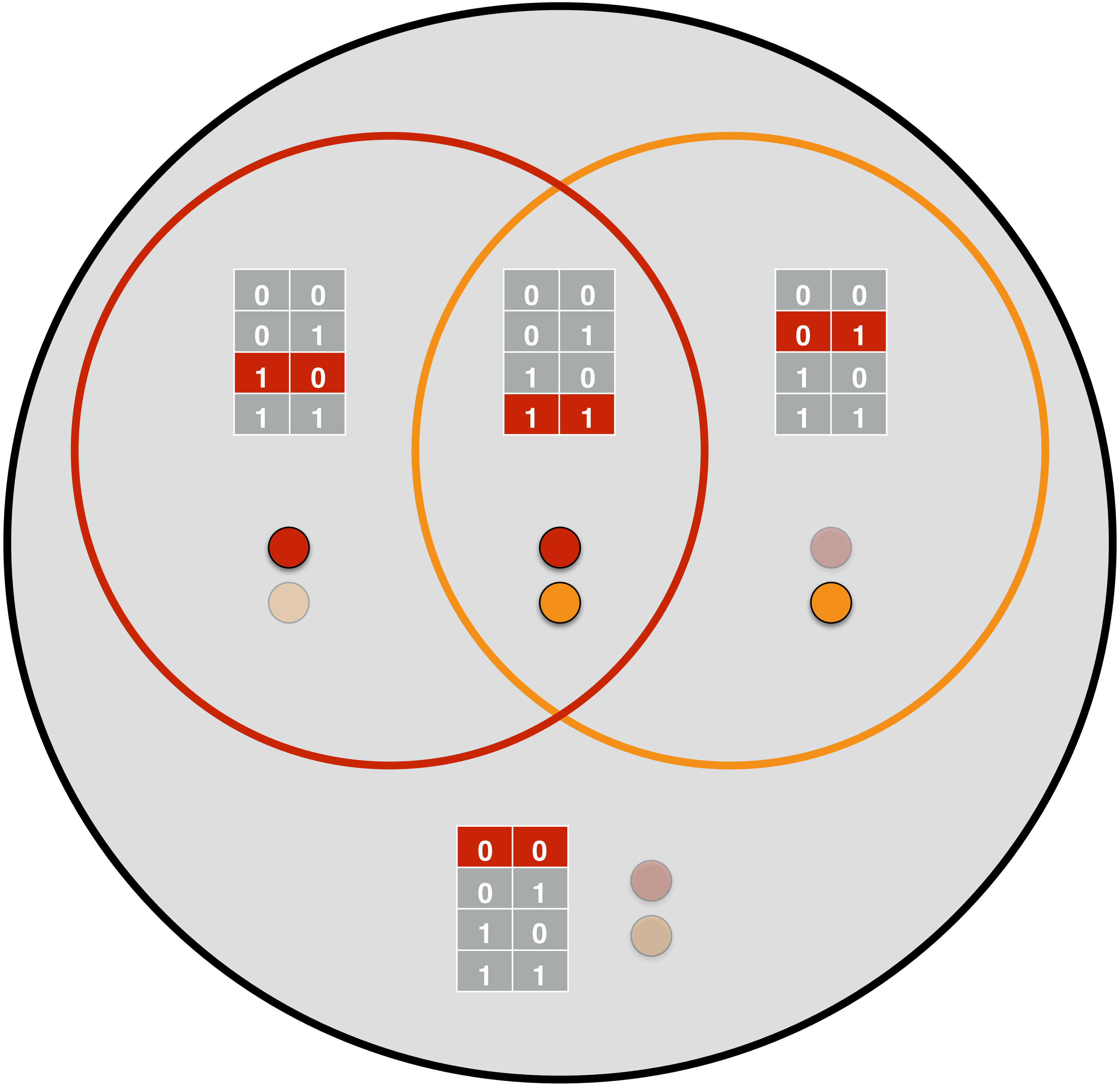
$$x \leq y \equiv x \wedge y = x$$

1. Show that, for any x, y, and z,

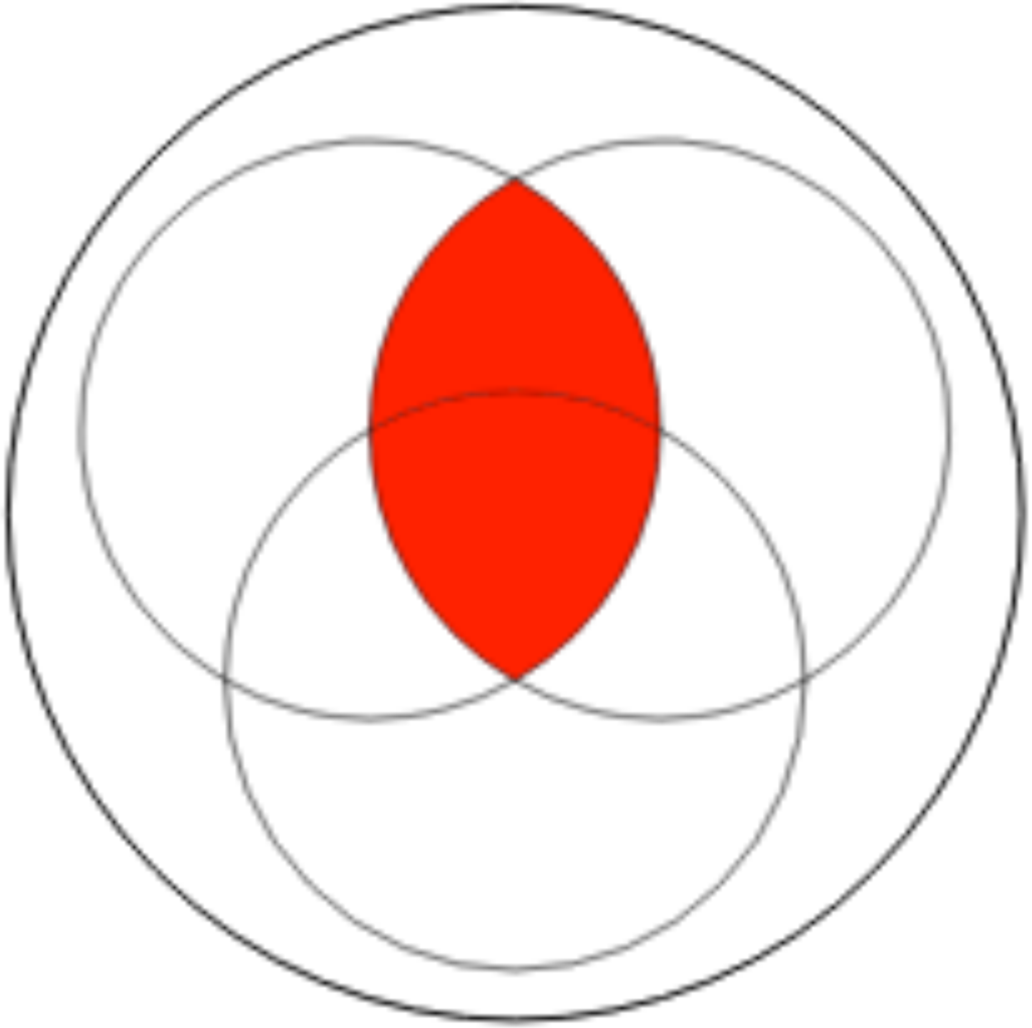
$$0 \leq x \text{ and } x \leq x \text{ and } x \leq 1$$
$$x \to y = \top \ \text{ iff } x \leq y$$
$$\text{if } x \leq y \text{ and } y \leq z \text{ then } x \leq z$$
$$\text{if } x \leq y \text{ and } y \leq x \text{ then } x = y$$
$$\text{if } x \leq y \text{ then } \neg y \leq \neg x$$

2. Show that, in any Boolean Algebra,

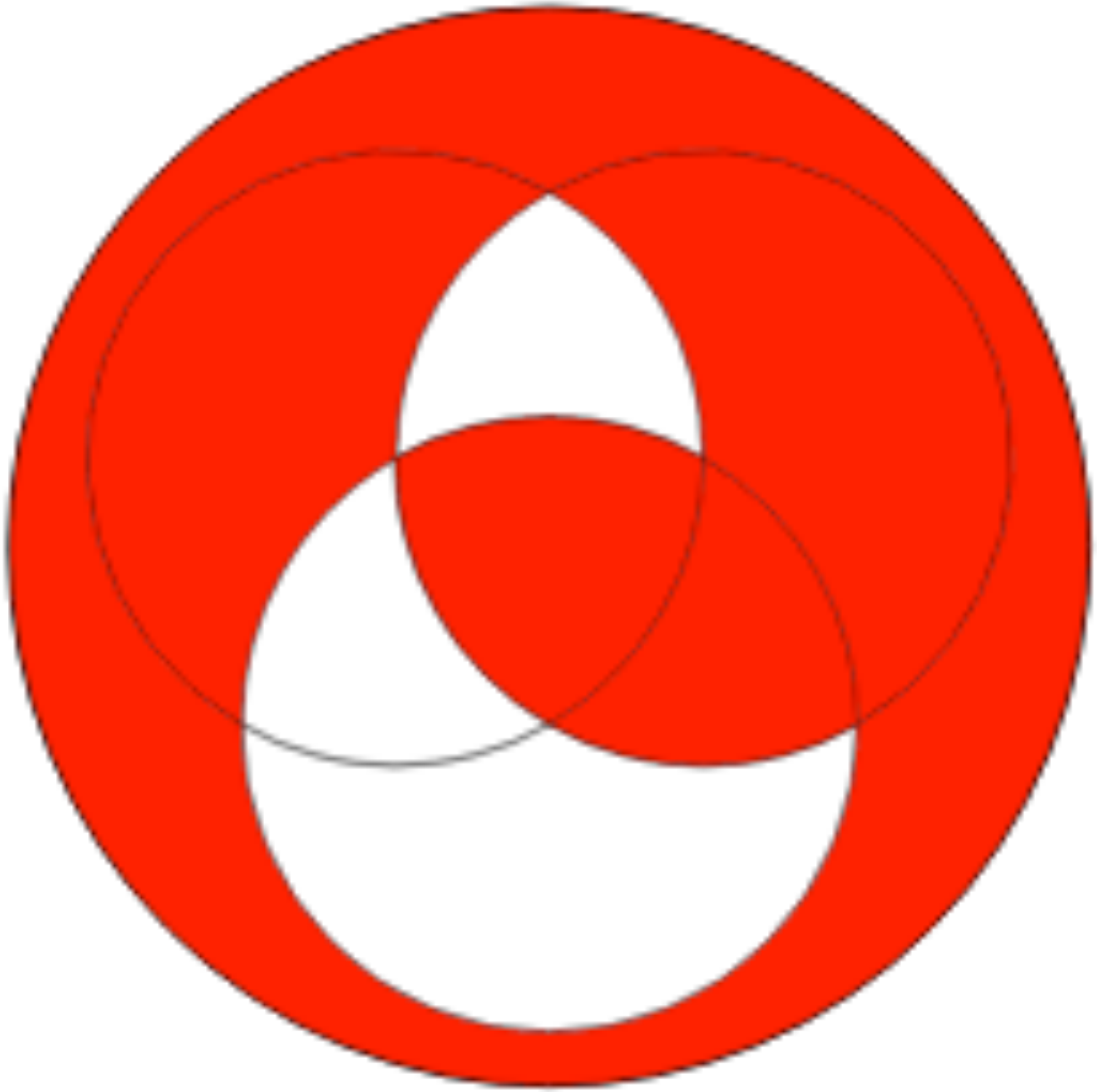
$$x \wedge y = x \text{ iff } x \vee y = y$$


0 0
0 1
1 0
1 1
0 0
0 1
1 0
1 1
0 0
0 1
1 0
1 1
0 0
0 1
1 0
1 1

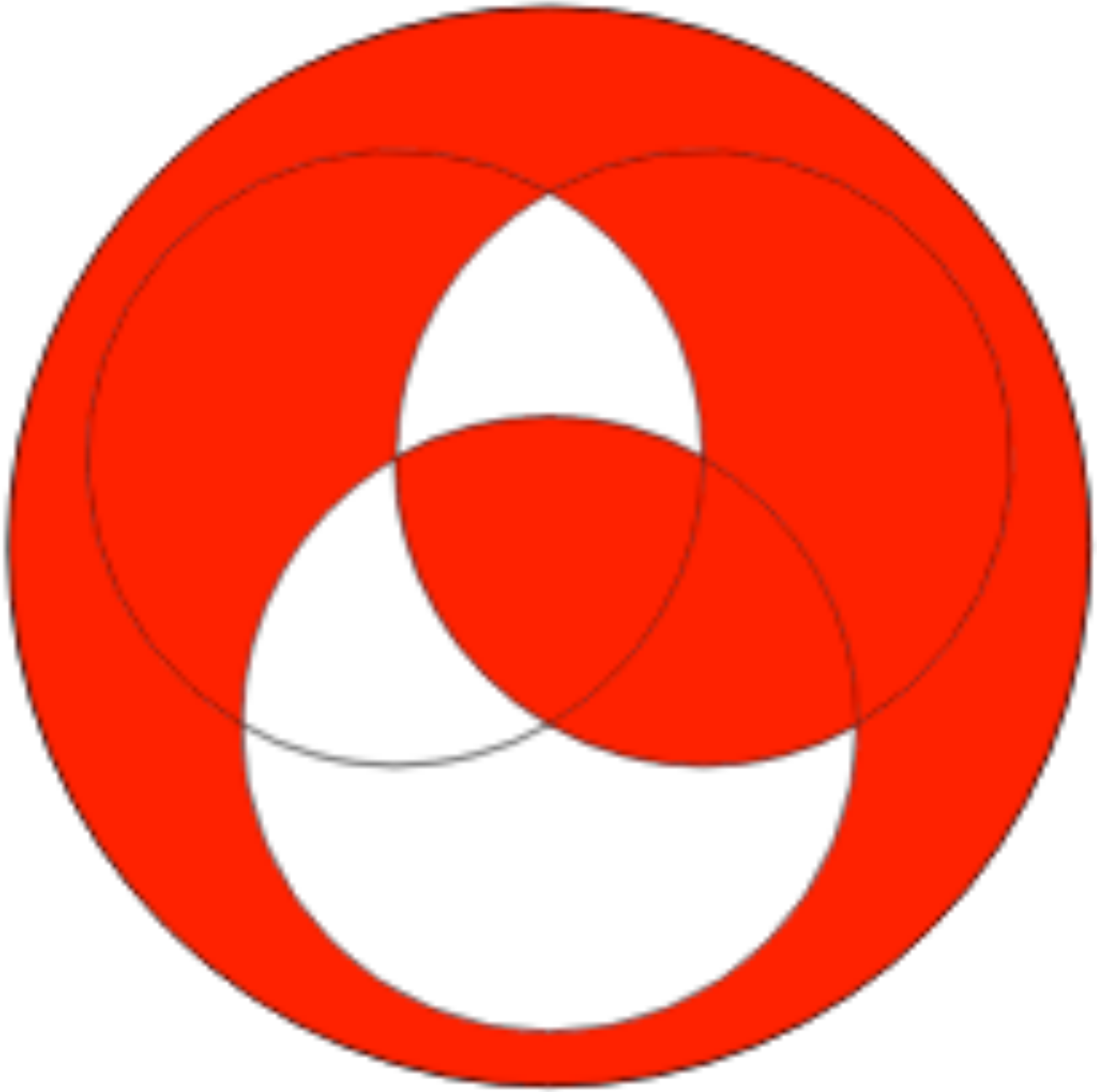
R && A



R	A	G	**R ∧ A**
0	0	0	**0**
0	0	1	**0**
0	1	0	**0**
0	1	1	**0**
1	0	0	**0**
1	0	1	**0**
1	1	0	**1**
1	1	1	**1**



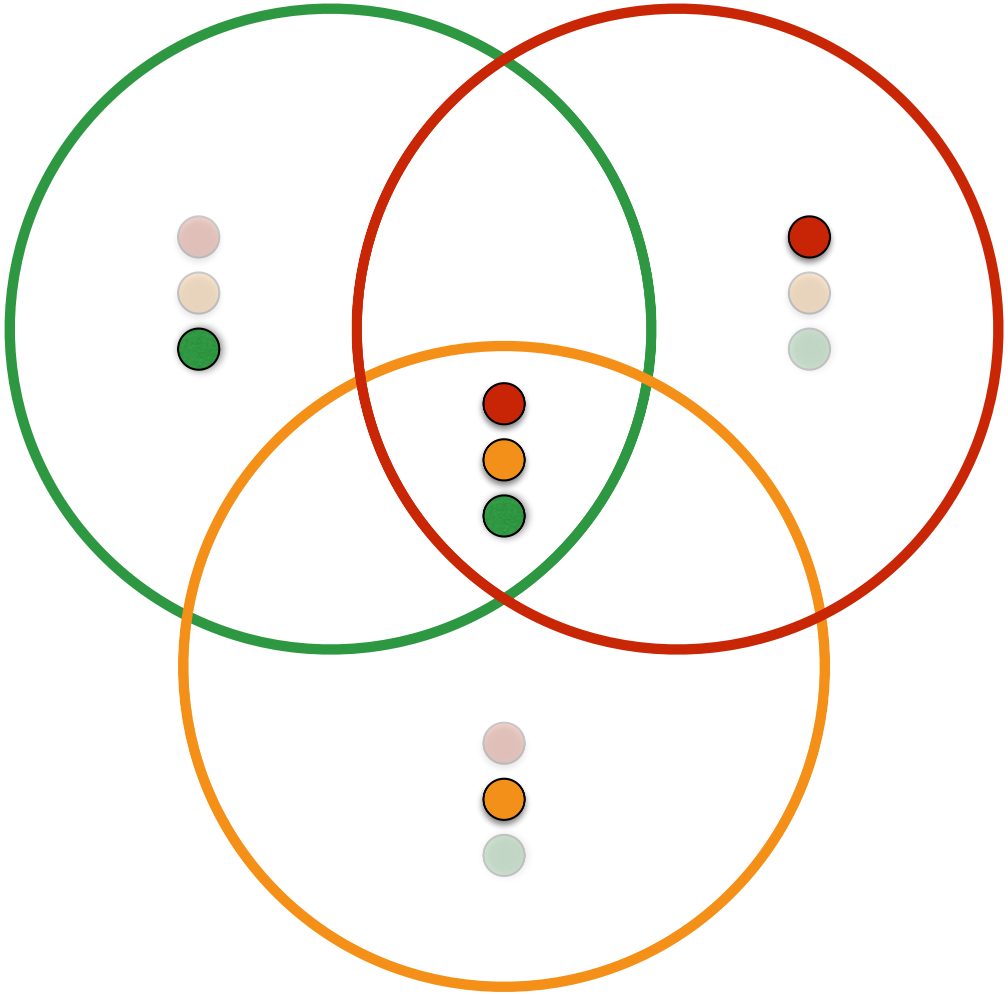
R	A	G	??
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

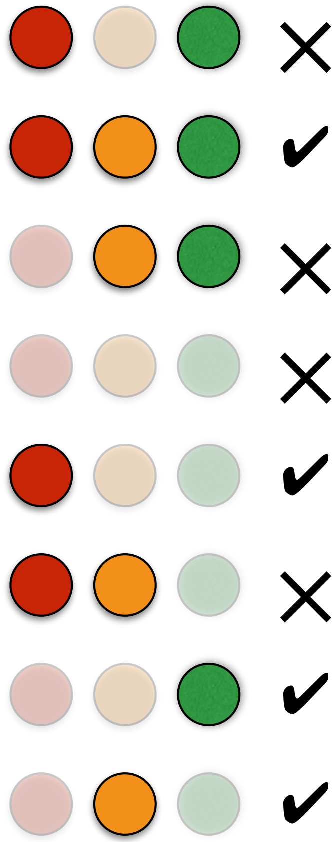


R	A	G	**??**
0	0	0	**1**
0	0	1	**0**
0	1	0	**1**
0	1	1	**1**
1	0	0	**1**
1	0	1	**0**
1	1	0	**0**
1	1	1	**1**

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$

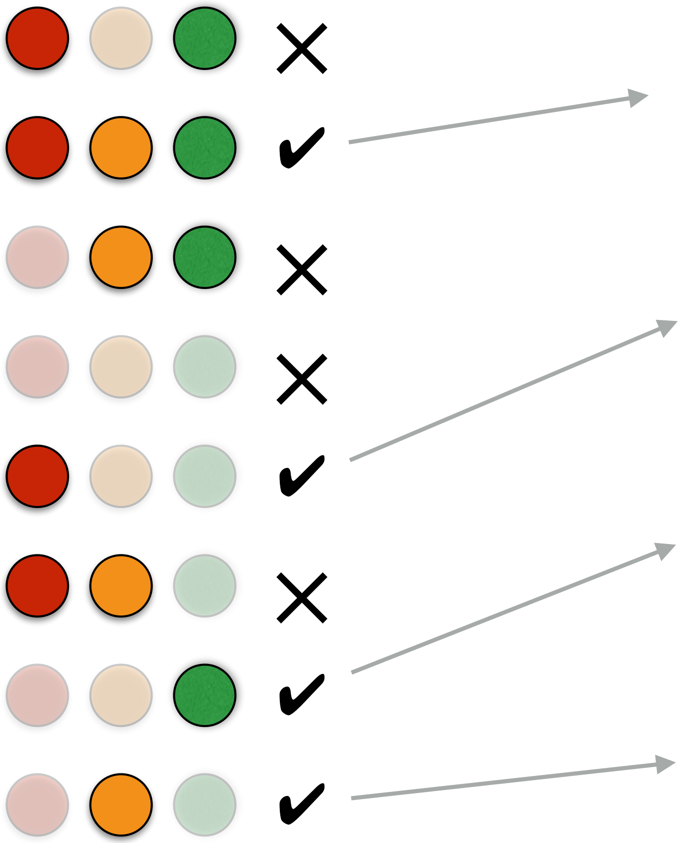
The meaning of an expression is the set of states in which it is true.





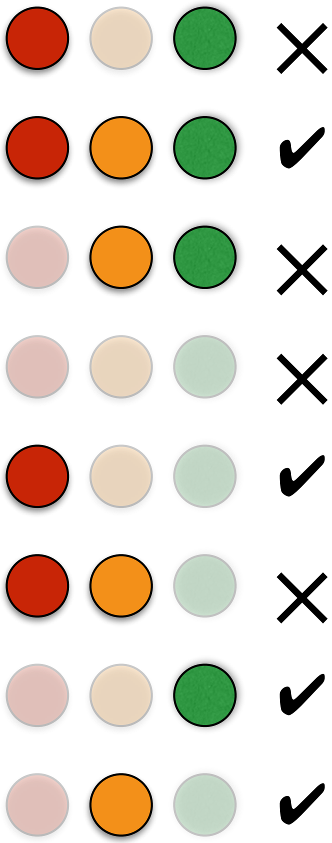
$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$

Disjunctive Normal Form (DNF)



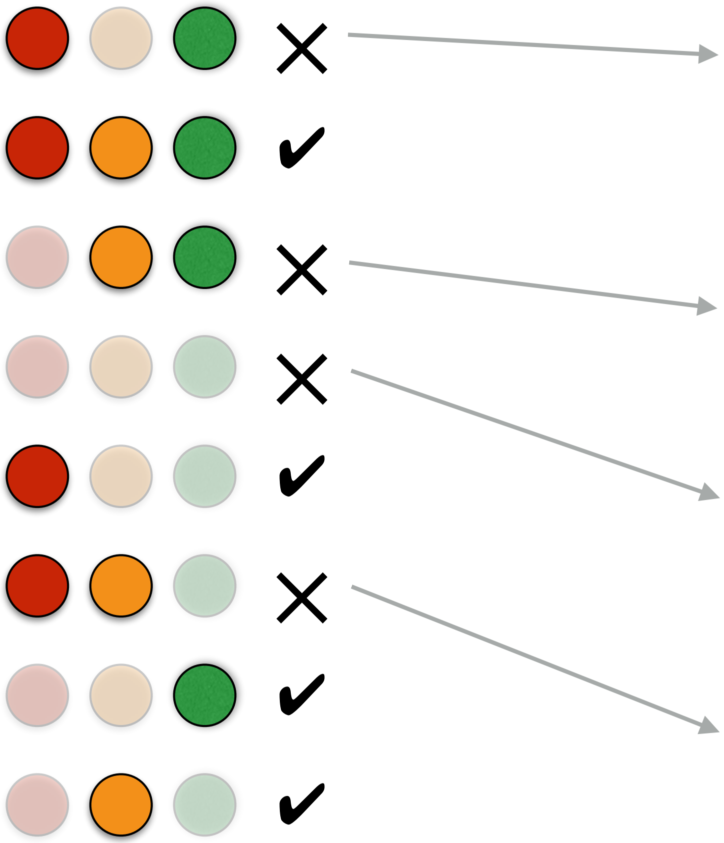
$$R(x) \wedge A(x) \wedge G(x)$$
$$\vee$$
$$R(x) \wedge \neg A(x) \wedge \neg G(x)$$
$$\vee$$
$$\neg R(x) \wedge \neg A(x) \wedge G(x)$$
$$\vee$$
$$\neg R(x) \wedge A(x) \wedge \neg G(x)$$

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$



$$\neg\Big(R(x) \wedge \neg A(x) \wedge G(x) \\ \vee \\ \neg R(x) \wedge A(x) \wedge G(x) \\ \vee \\ \neg R(x) \wedge \neg A(x) \wedge \neg G(x) \\ \vee \\ R(x) \wedge A(x) \wedge \neg G(x) \Big)$$

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$



$$\neg\big(R(x) \wedge \neg A(x) \wedge G(x)\big)$$

$$\wedge$$

$$\neg\big(\neg R(x) \wedge A(x) \wedge G(x)\big)$$

$$\wedge$$

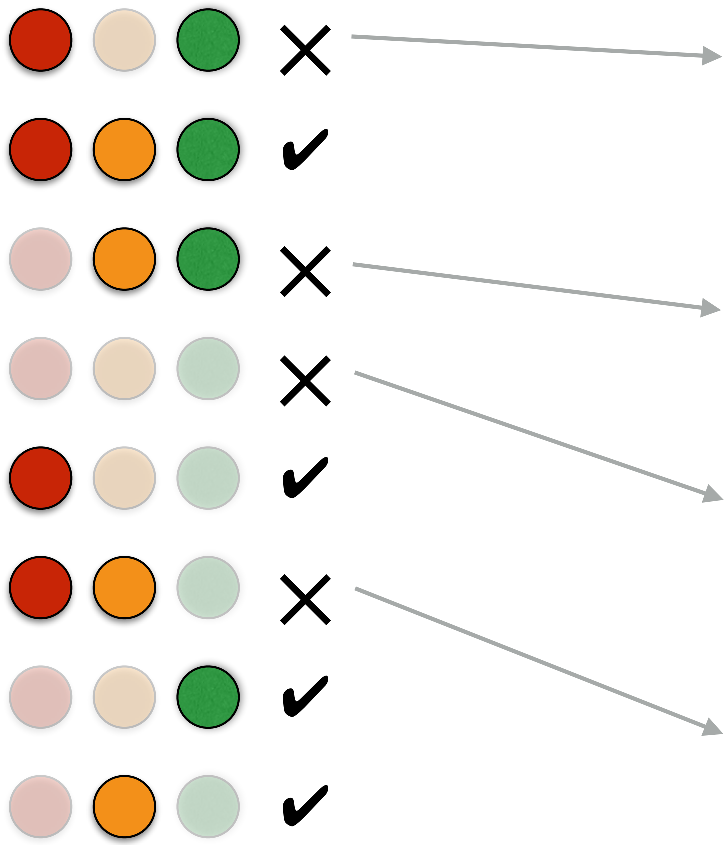
$$\neg\big(\neg R(x) \wedge \neg A(x) \wedge \neg G(x)\big)$$

$$\wedge$$

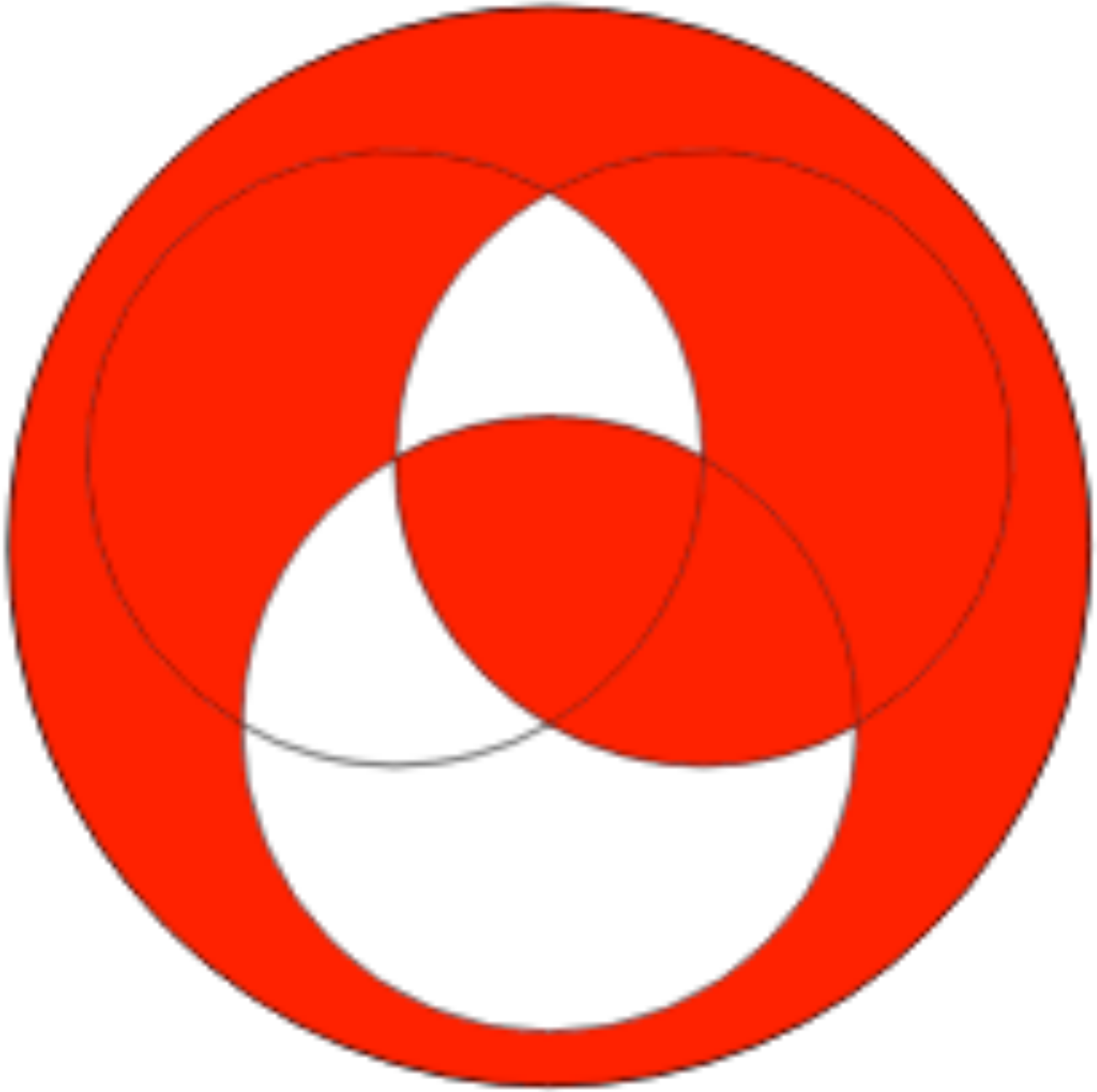
$$\neg\big(R(x) \wedge A(x) \wedge \neg G(x)\big)$$

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$

Conjunctive Normal Form (CNF)

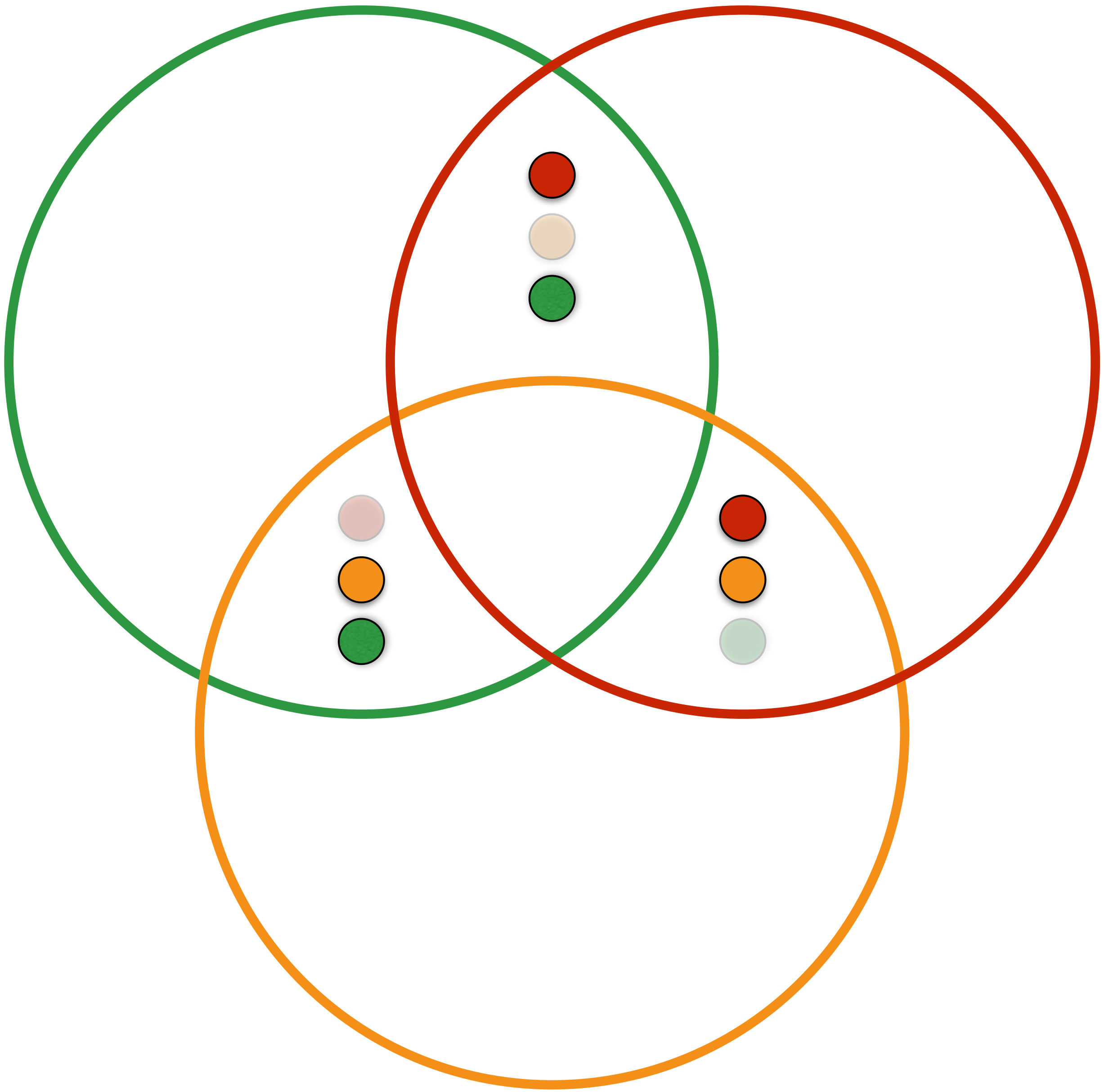


$$\big(\neg R(x) \lor A(x) \lor \neg G(x)\big)$$
$$\land$$
$$\big(R(x) \lor \neg A(x) \lor \neg G(x)\big)$$
$$\land$$
$$\big(R(x) \lor A(x) \lor G(x)\big)$$
$$\land$$
$$\big(\neg R(x) \lor \neg A(x) \lor G(x)\big)$$



R	A	G	??
0	0	0	**1**
0	0	1	**0**
0	1	0	**1**
0	1	1	**1**
1	0	0	**1**
1	0	1	**0**
1	1	0	**0**
1	1	1	**1**

Exercise 2.2 Generate CNF for this subset



Exercise 2.3 Generate CNF for this subset