This course provides a first glimpse of the deep connections between computation and logic. We will focus primarily on the simplest non-trivial examples of logic and computation: propositional logic and finite-state machines.

In this first lecture we look at an example that introduces some ideas that we will explore further in later lectures, and introduce some notation which should become more familiar in due course.

The ancient problem of the Wolf, the Goose and the Corn provides a first example of a finite state system which we can describe and analyse using logic.

The farmer has a big stick and a small boat. She wants to take the Wolf, the Goose and the Corn from the West side of the river to the East side. When she is there, she can stop the Wolf from eating the Goose and the Goose from eating the Corn, but she can only take one load at a time in the boat.

A farmer has to get a wolf, a goose, and a sack of corn across a river.

She has a boat, which can only carry her and one other thing.

If the wolf and the goose are left together, the wolf will eat the goose.

If the goose and the corn are left together, the goose will eat the corn.

How does she do it?

4

It is not safe for her to leave the Wolf with the Goose, or the Goose with the Corn.

She can leave the Wolf with the Corn.

We introduce some basic propositions to describe the state.

Here the following propositions are true

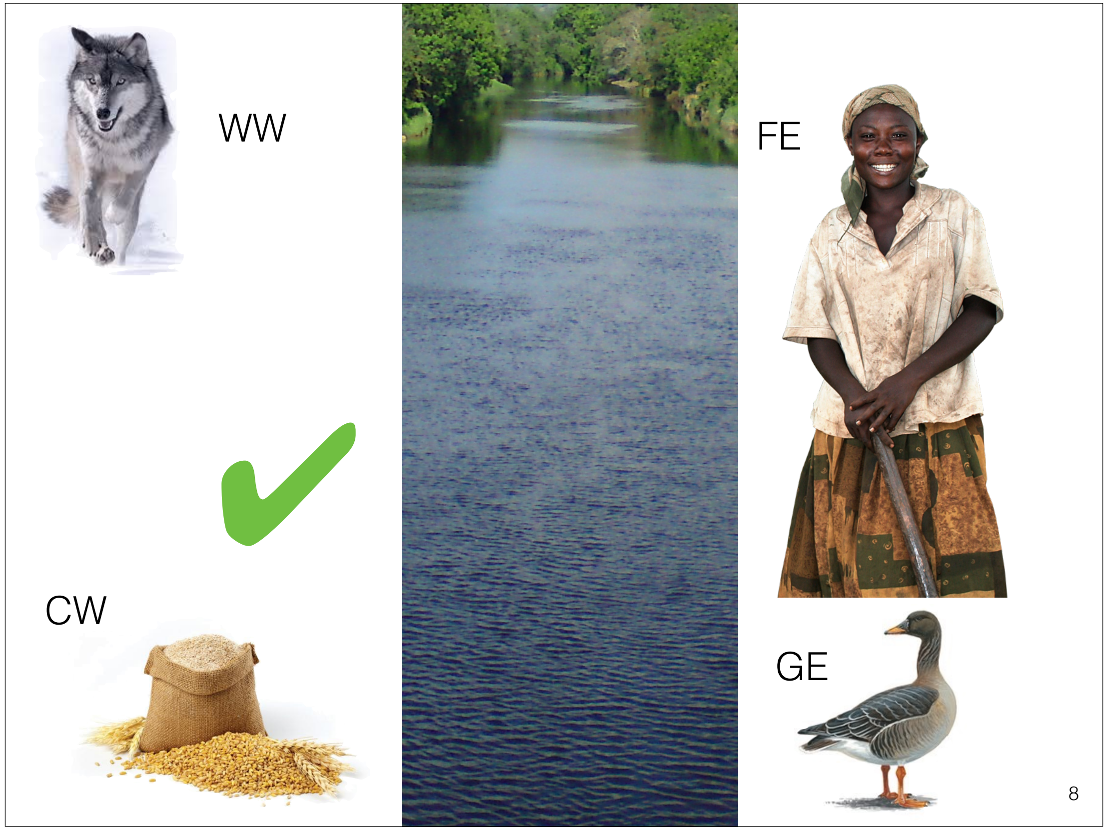
WW The Wolf is on the West

GW The Goose is on the West

CW The Corn is on the West

FE The Farmer is on the East

This is not a safe state.



Here the following propositions are true

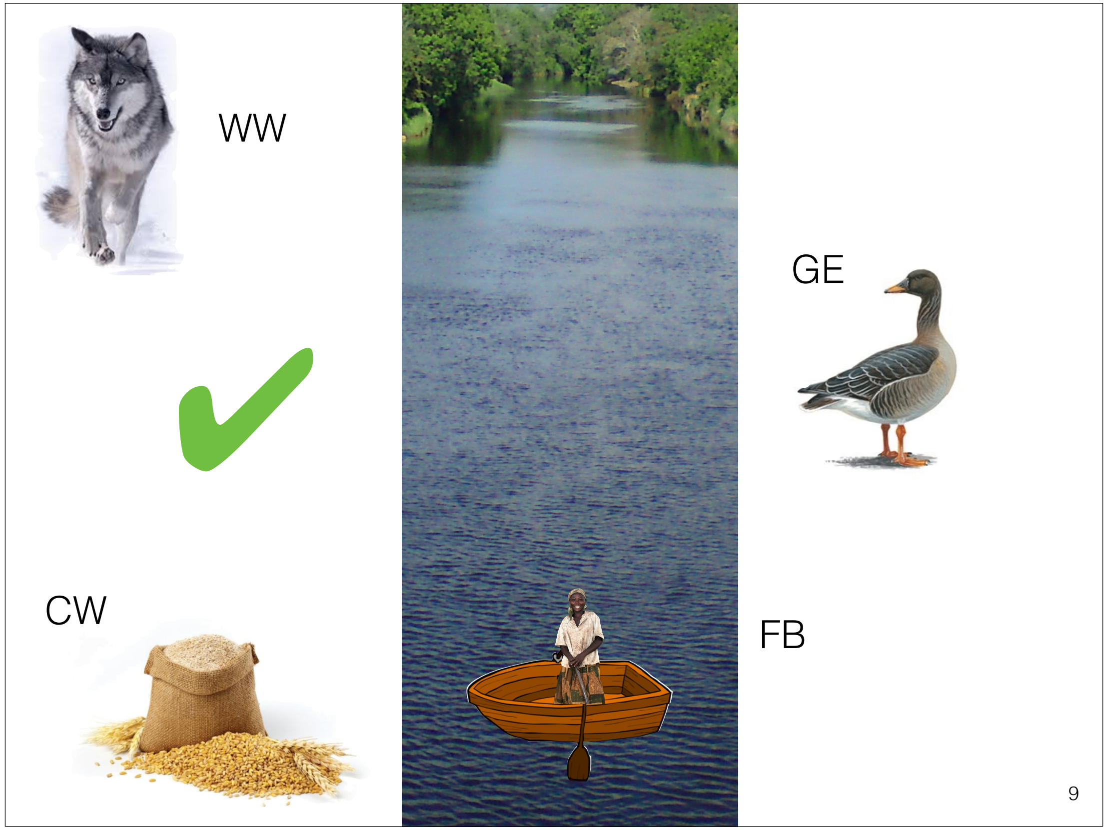
WW The Wolf is on the West

GE The Goose is on the East

CW The Corn is on the West

FE   The Farmer is on the East

This is a safe state.



Here the following propositions are true

WW The Wolf is on the West

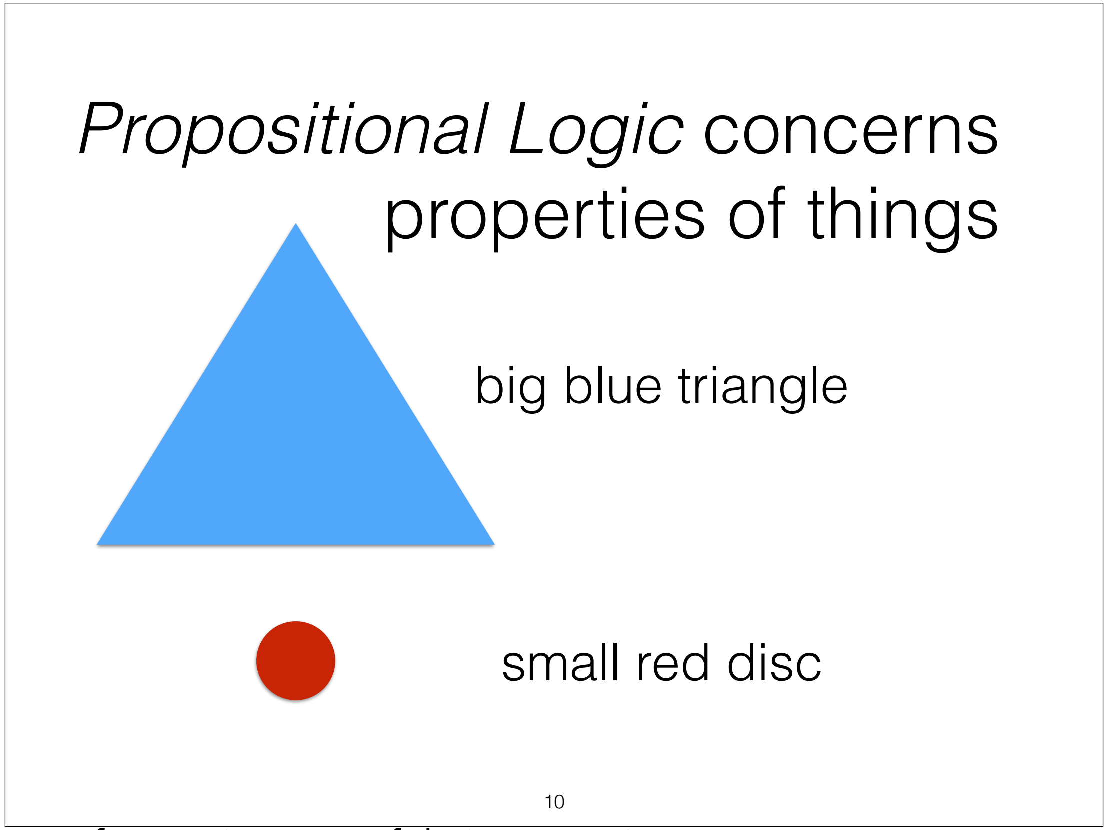
GE The Goose is on the East

CW The Corn is on the West

FB The Farmer is in the Boat

This is a safe state.

We will use logic to describe the safe, legal, possible states.

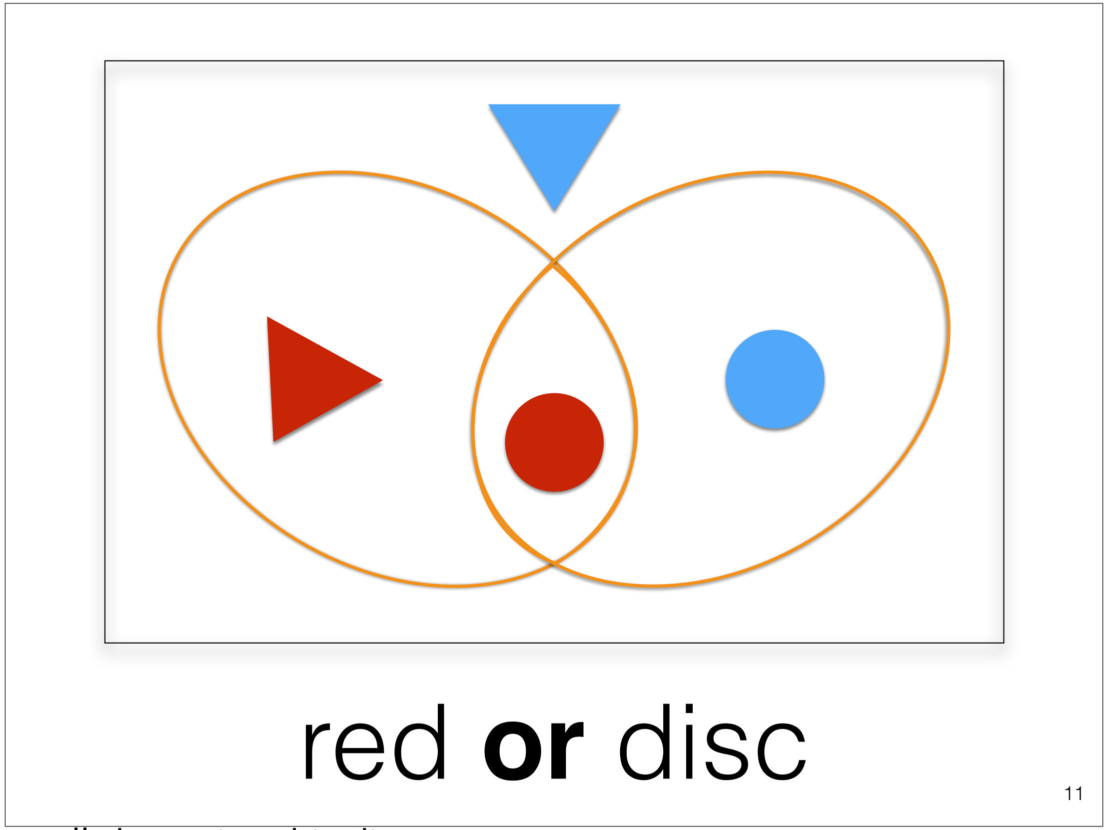


We will use logic to describe sets of states in terms of their properties.

However it is simpler to start by using logic to describe sets of things in terms of their properties.

For this part of the lecture, we consider a very simple 'world', where everything is either red or blue, either big or small, and either a triangle or a disc.

Moreover, there is one, and only one thing of each type: only one big blue triangle, only one small red disc, and so on …

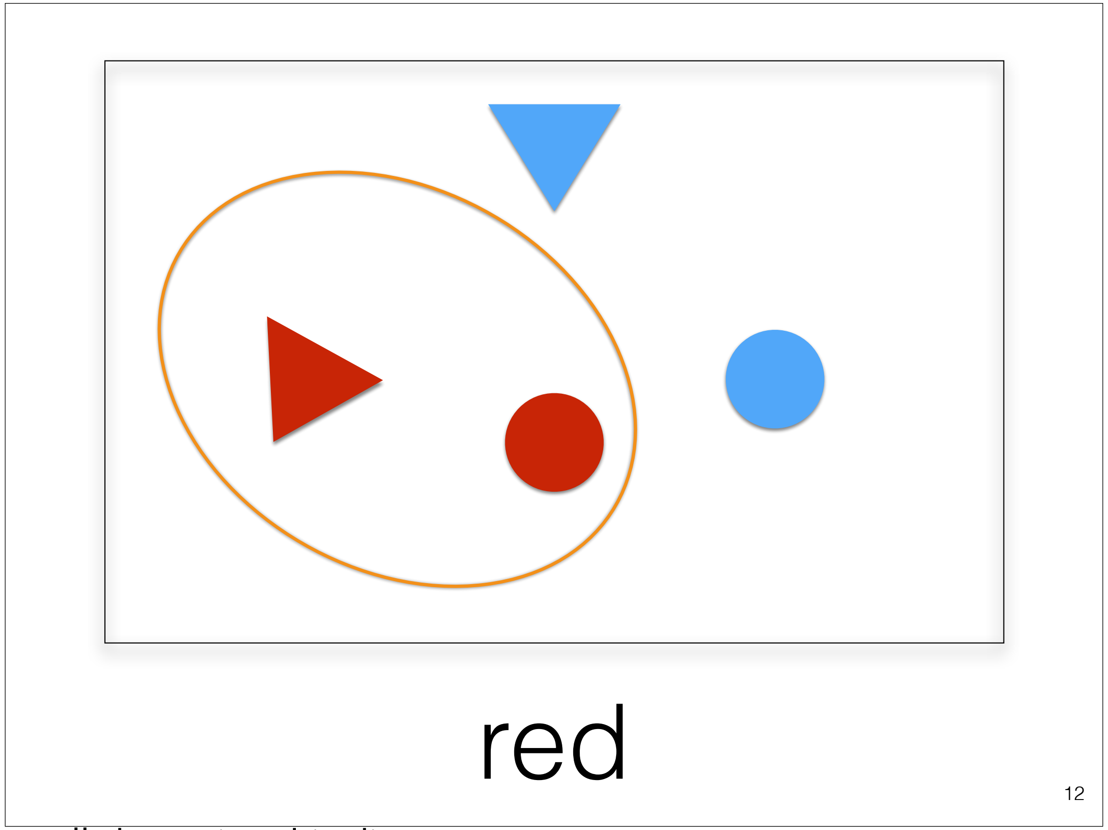


There are only four small things, all shown inn this diagram.

The diagram also includes two circles, representing sets of things.

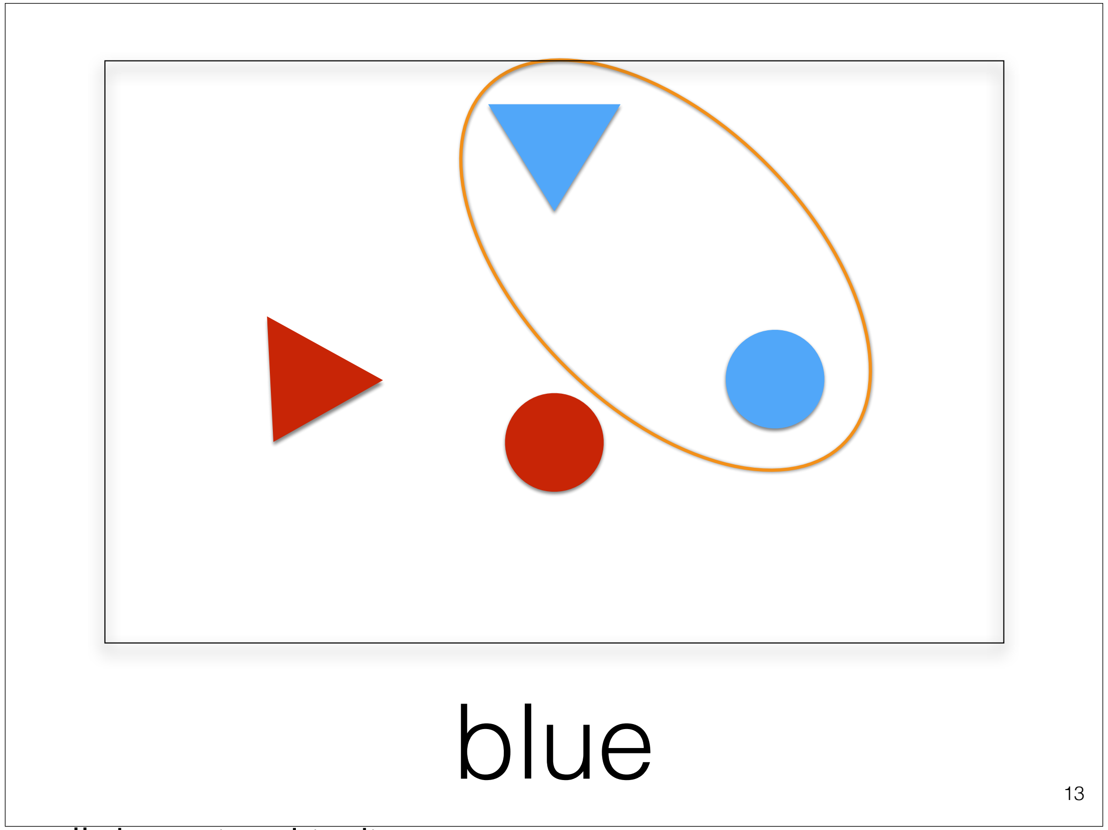
Each of these sets is defined by a property.

One represents the set of small red things, the other represents the set of small discs.



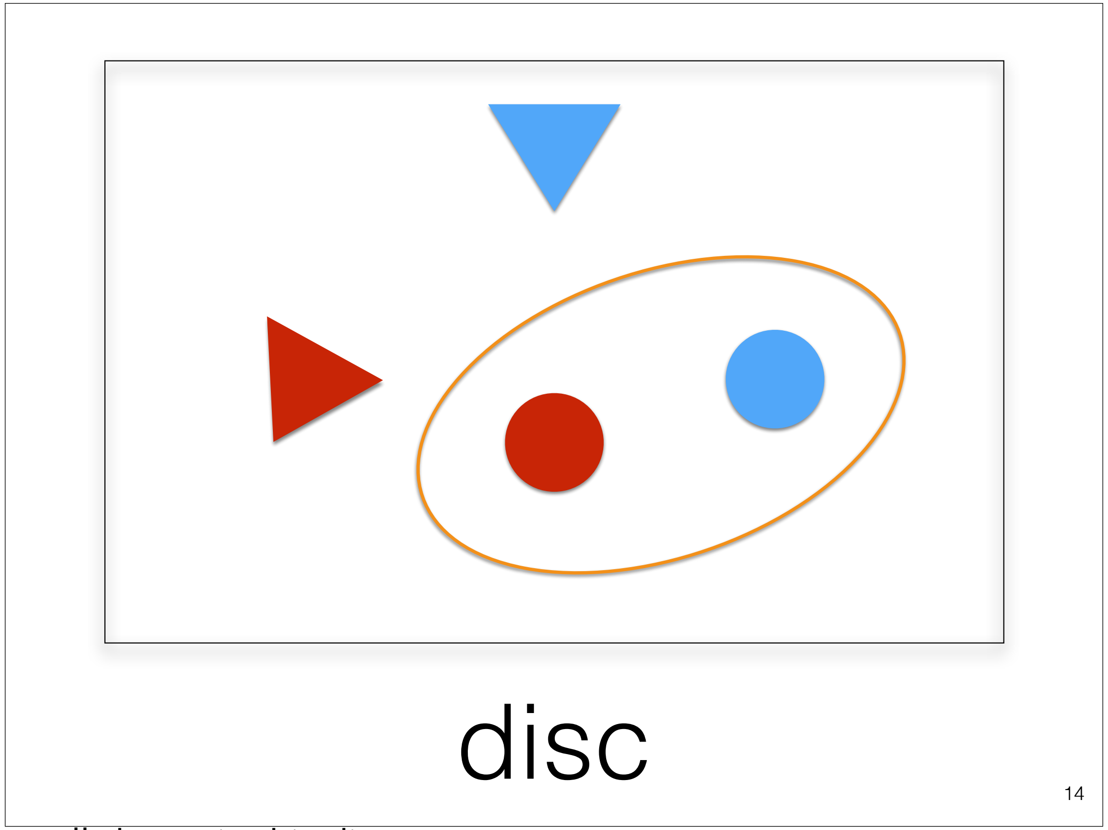
There are only four small things, all shown inn this diagram.

The circle represents the set of small red things.



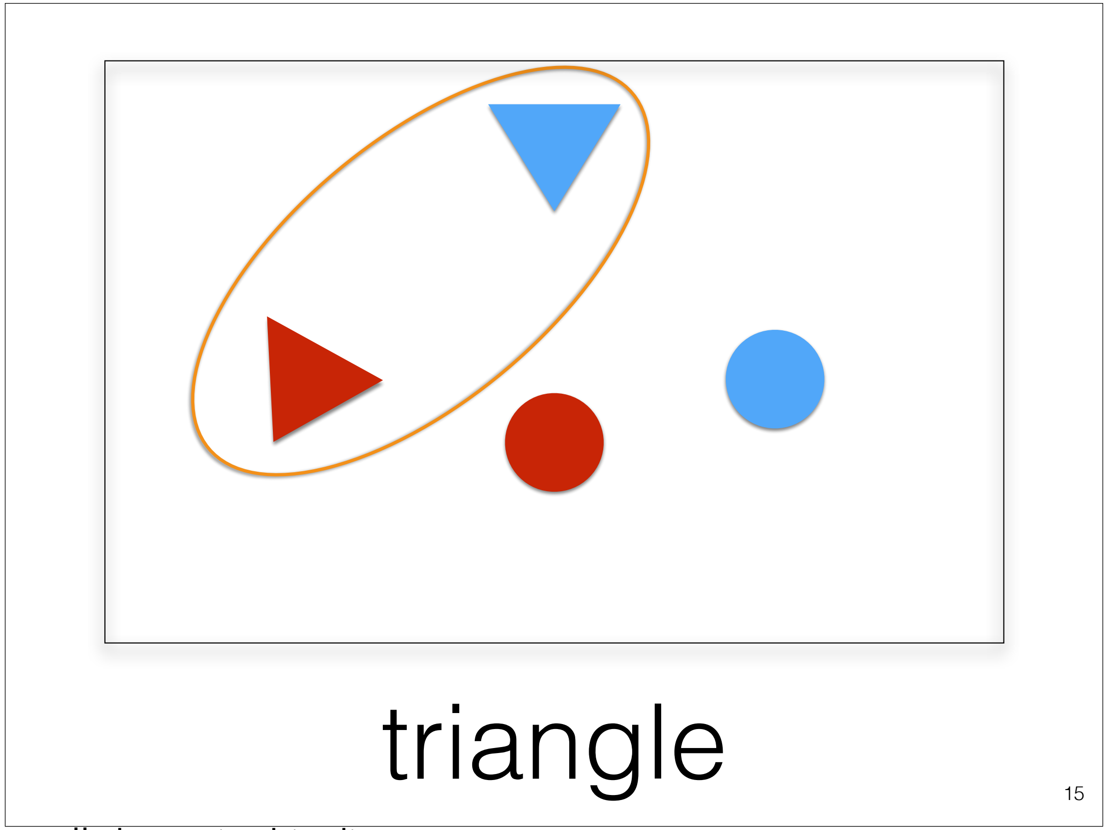
There are only four small things, all shown inn this diagram.

The circle represents the set of small blue things.



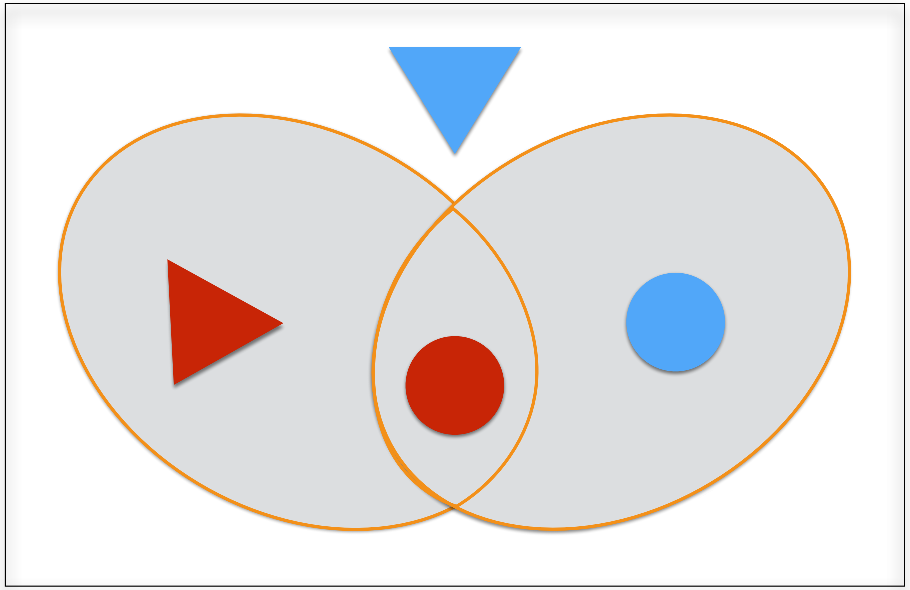
There are only four small things, all shown in this diagram.

The circle represents the set of small discs.



There are only four small things, all shown in this diagram.

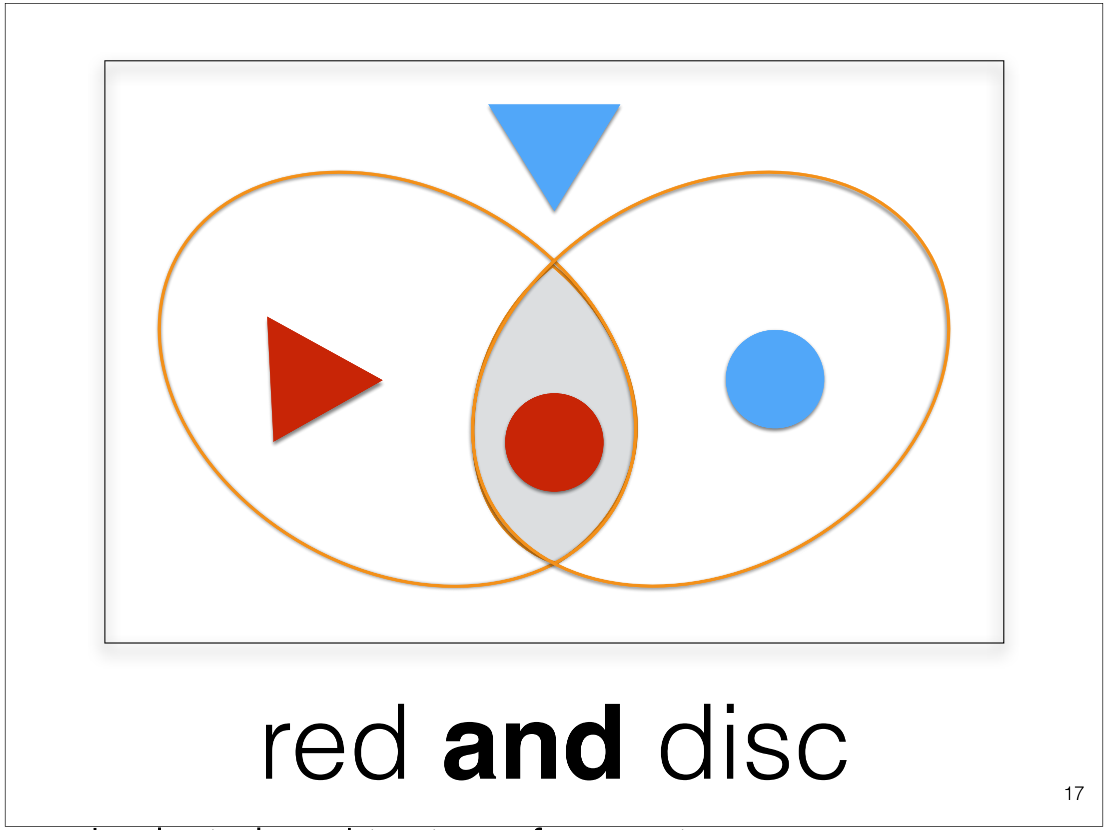
The circle represents the set of small triangles.



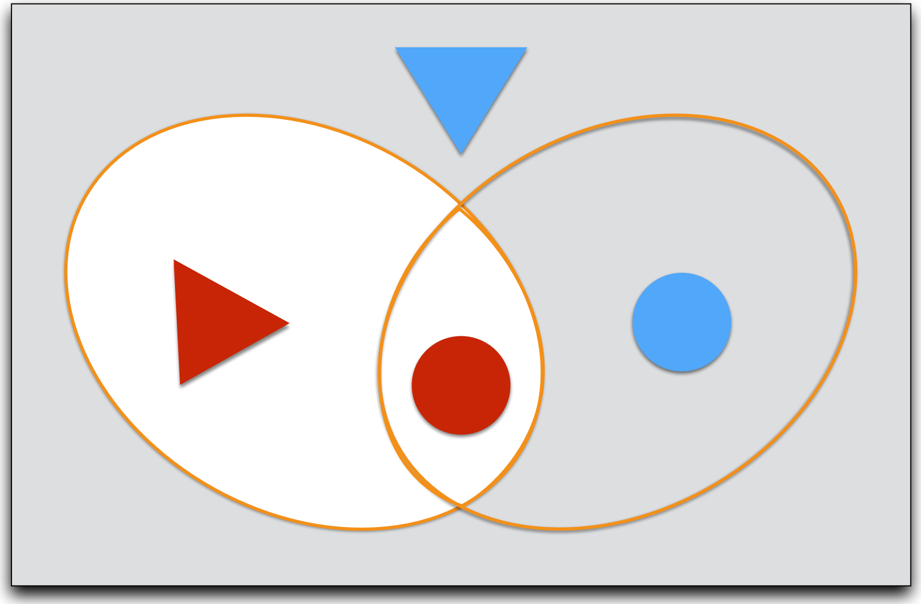
red **or** disc

16

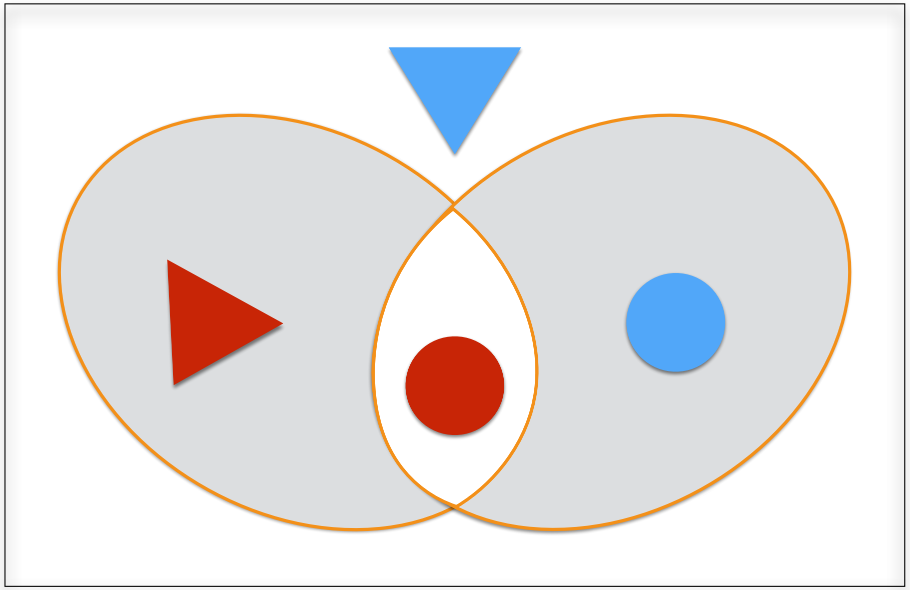
Regions of the diagram correspond to logical combinations of properties.



Regions of the diagram correspond to logical combinations of properties



not red



red **xor** disc

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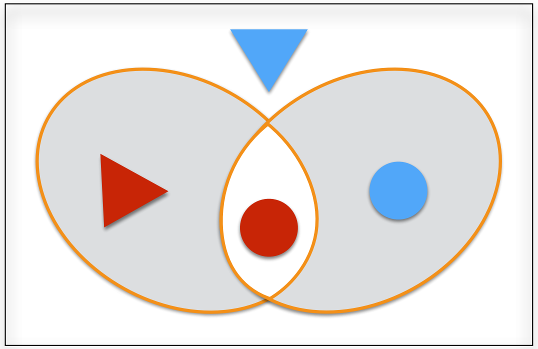
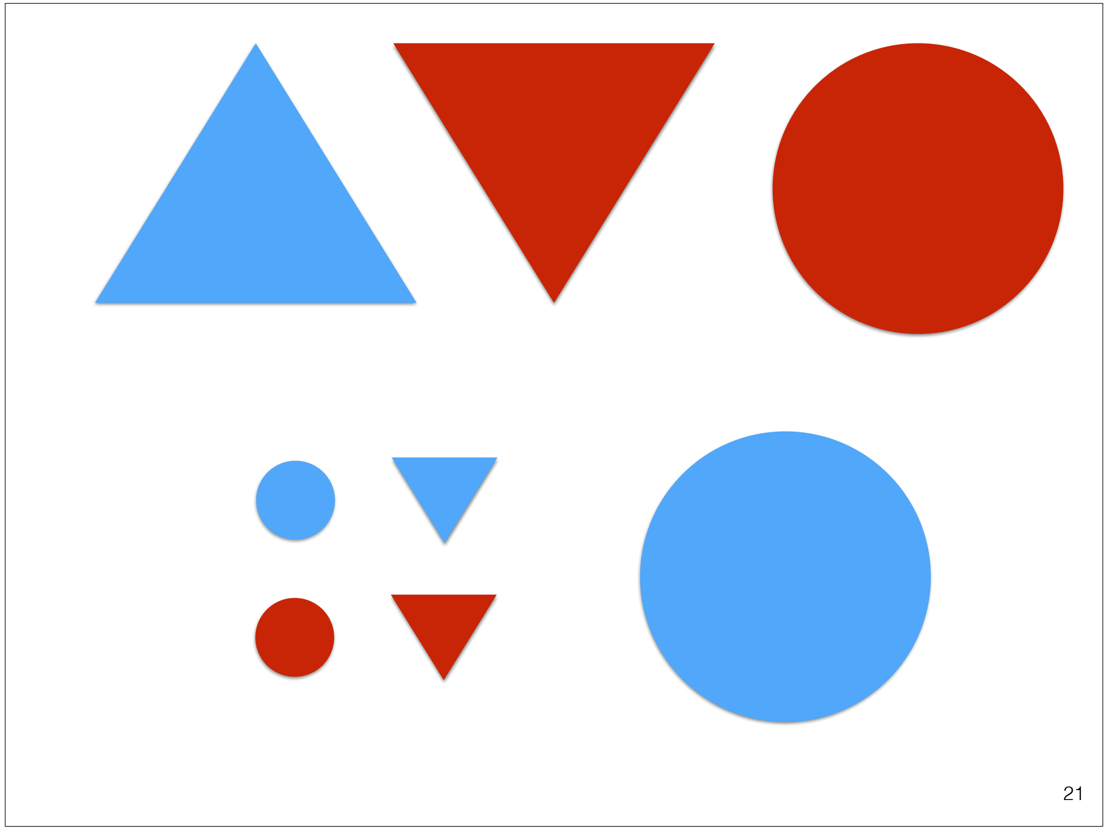


(red **or** disc) **and**
not (red **and** disc)
=
red **xor** disc

20



If everything is

either red or blue (not red)

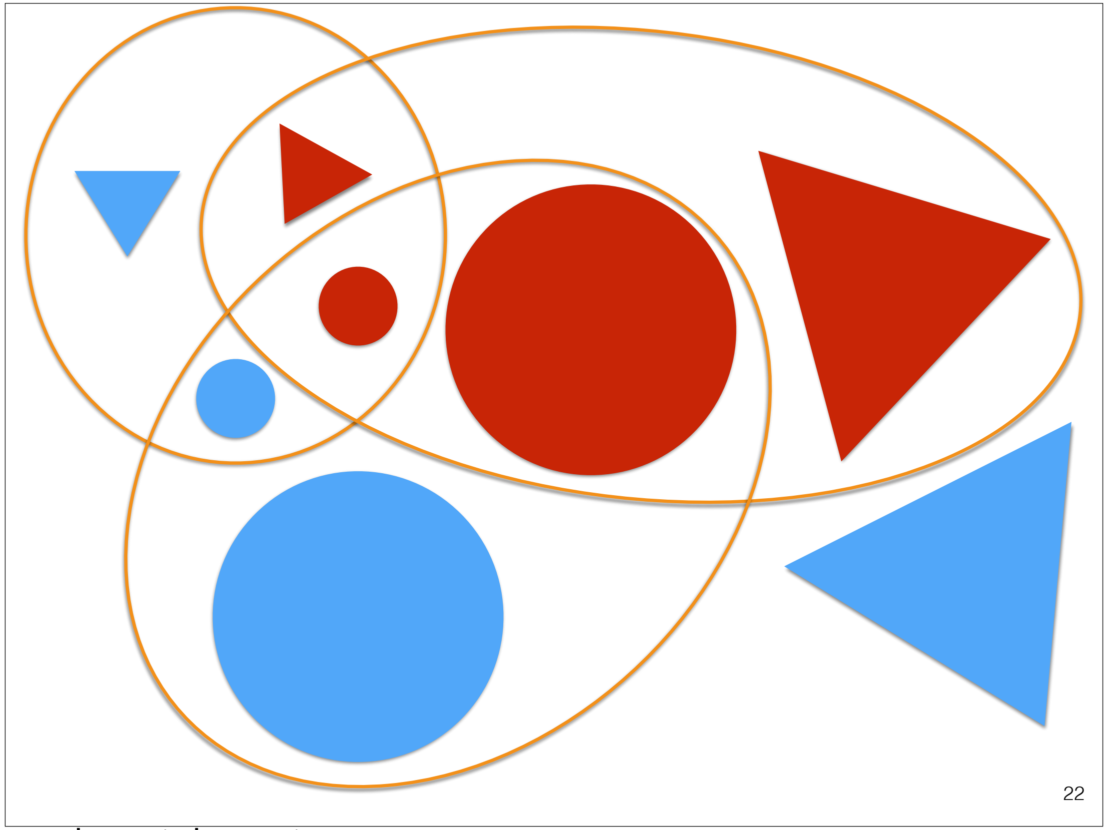
and

either small or big (not small)

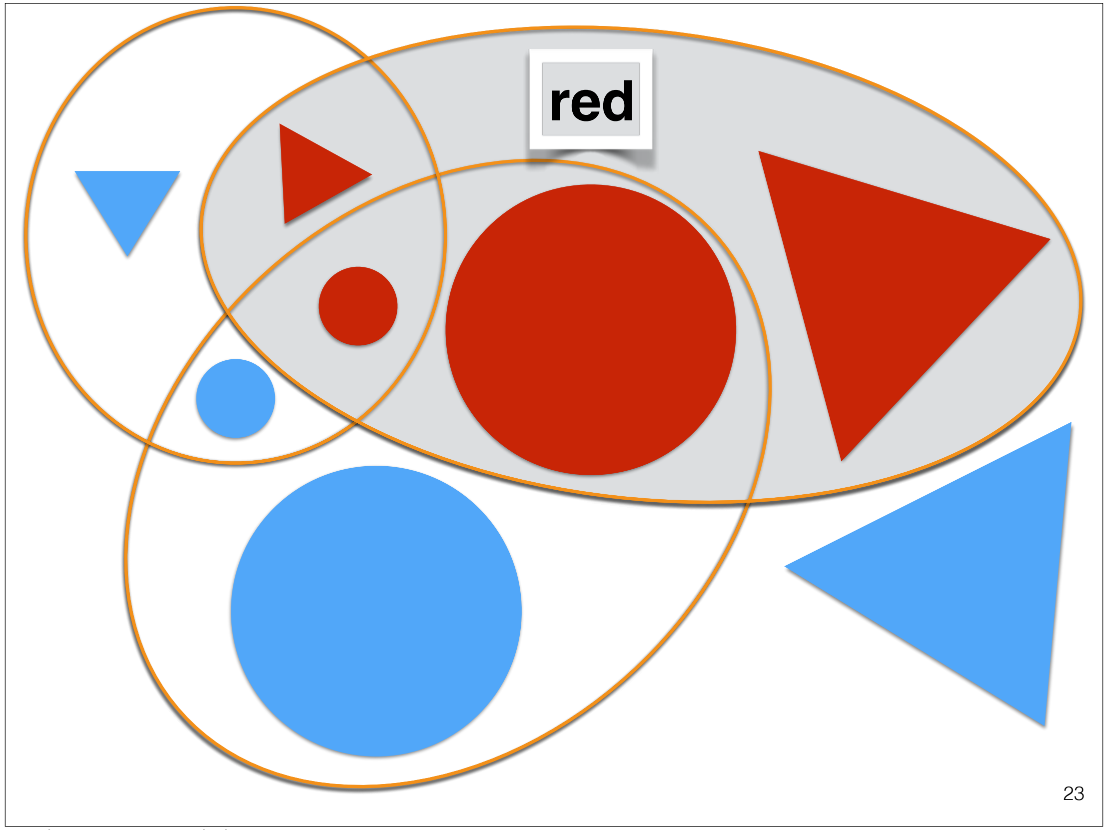
and

either disc or triangle (not disc)

then we have 8 = 2 x 2 x 2 possible combinations of three Boolean properties.



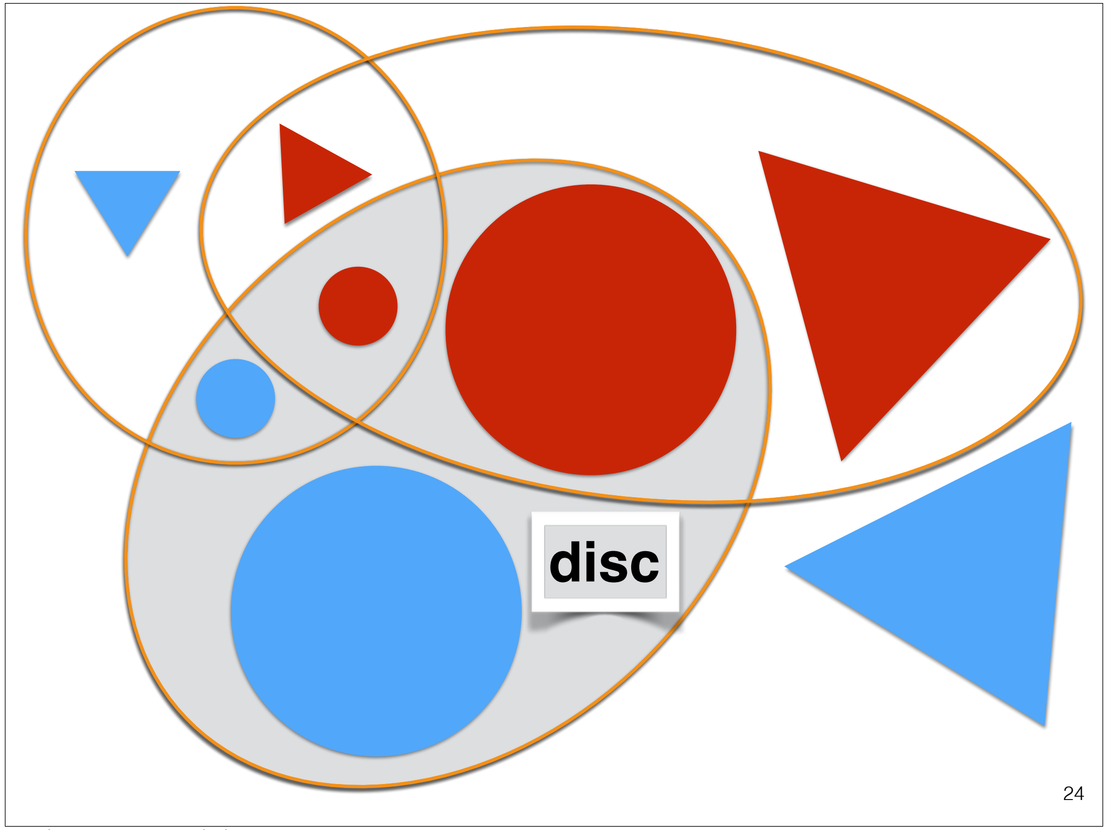
The corresponding *Venn diagram* has eight regions.



Each circle corresponds to a basic proposition.

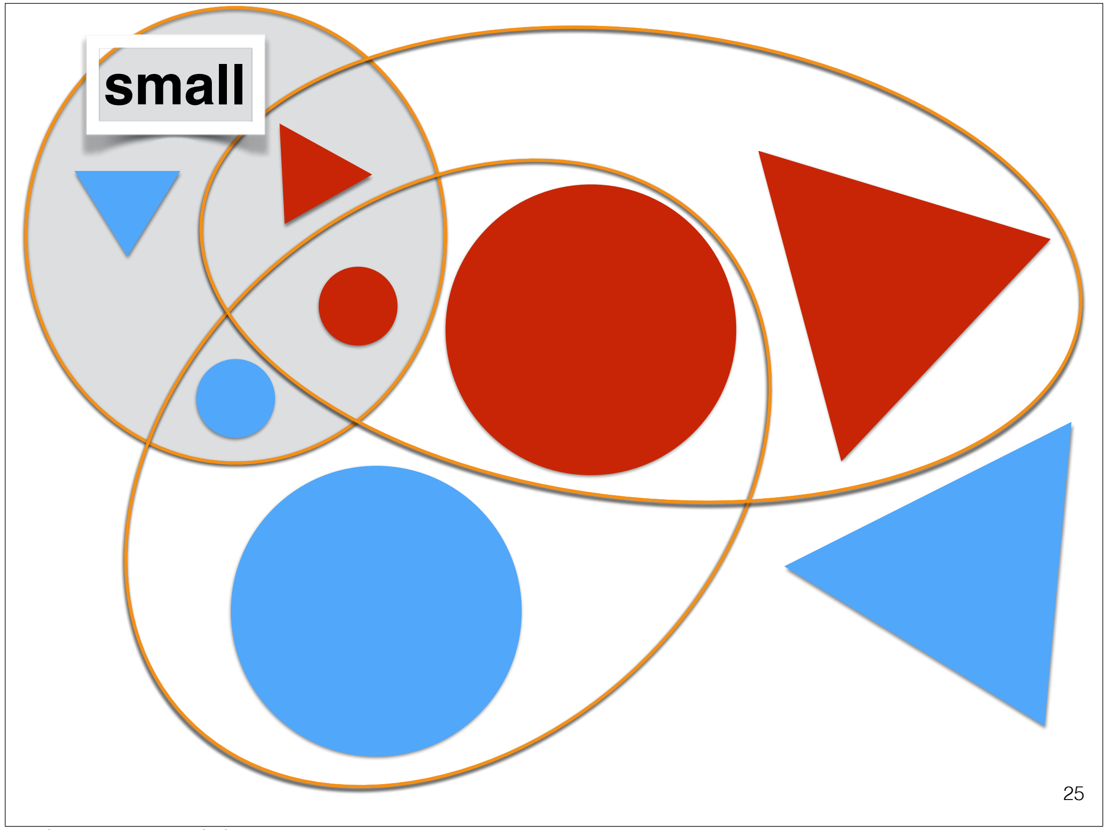
red

Each circle includes four of the eight regions



Each circle corresponds to a basic proposition.

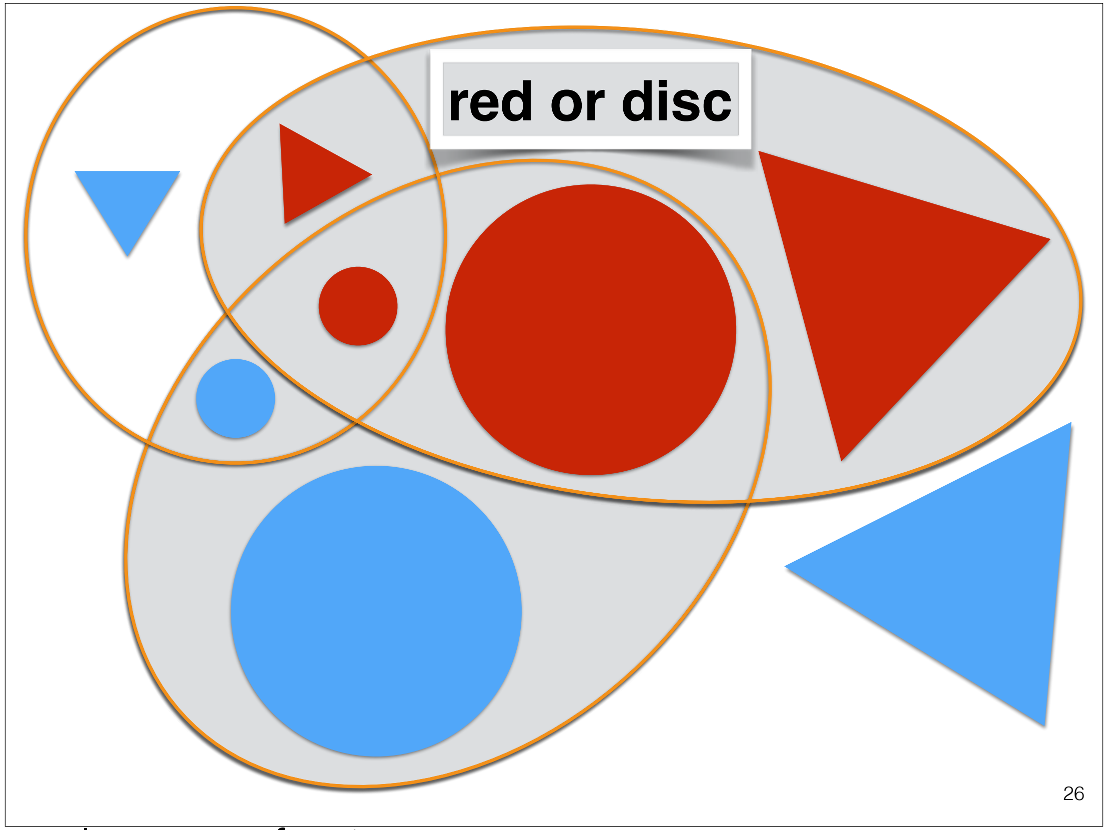
disc



Each circle corresponds to a basic proposition.

small

Each circle includes

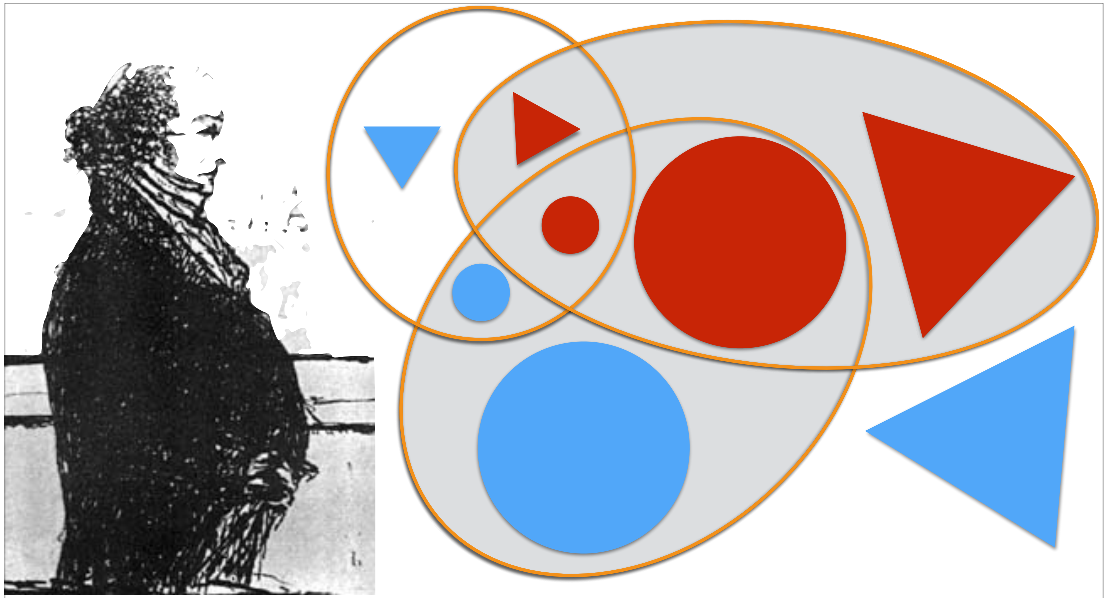


A complex proposition corresponds to a set of regions.

red or disc

This example includes six of the eight regions

The blue triangles, which are not red and not disc, are excluded.



not (red or disc) iff (not red and not disc)

Augustus de Morgan (1806 - 1871)

27

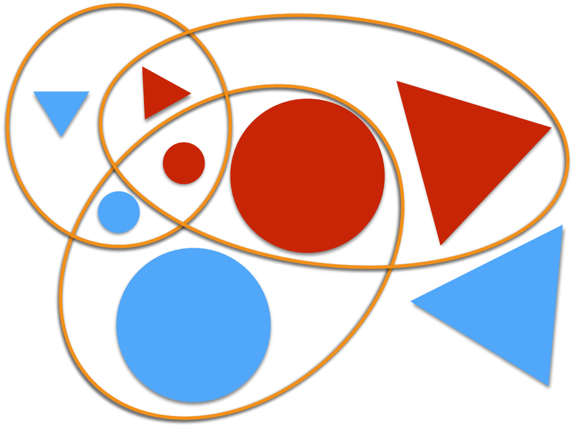
A complex proposition corresponds to a set of regions.

red or disc

This example includes six of the eight regions

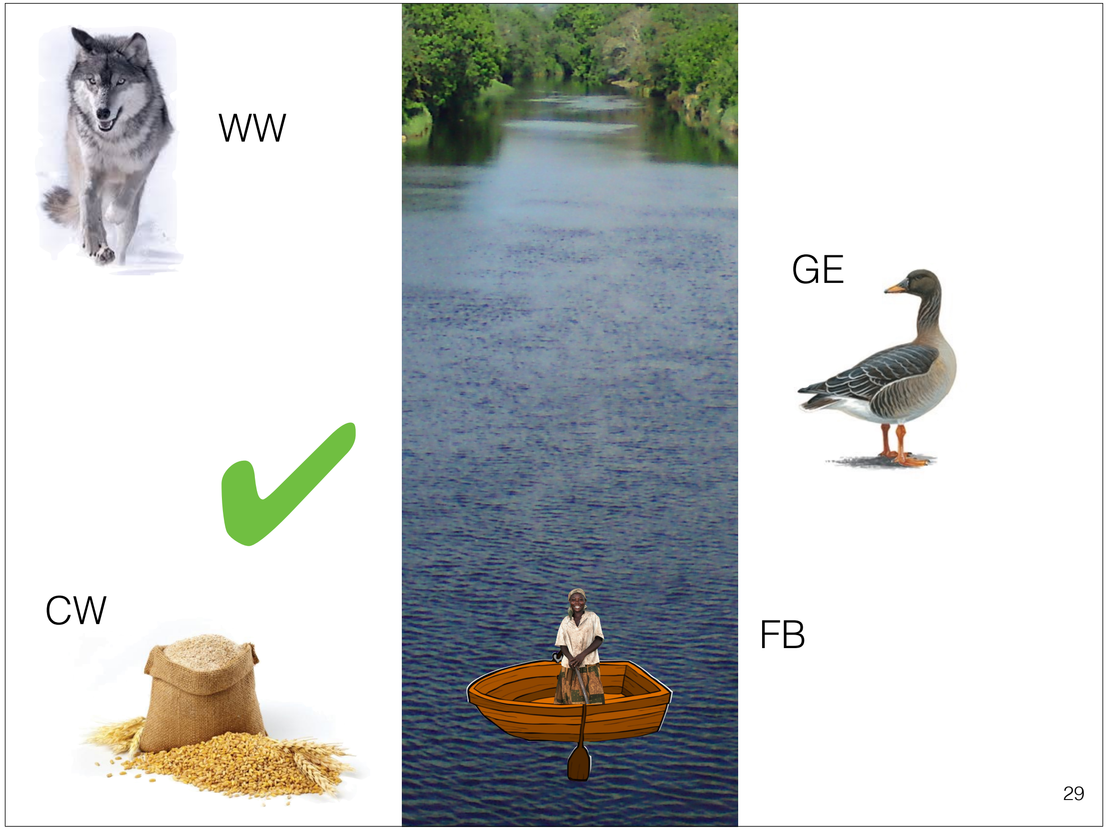
The blue triangles, which are not red and not disc, are excluded.

Exercise 1.1



There are 8 regions in the diagram. How many subsets of this set of 8 regions are there?

Given any subset of the eight regions can you write a complex proposition to which it corresponds
(using **and**, **or**, and **not** as connectives)?



Here the following propositions are true

WW The Wolf is on the West

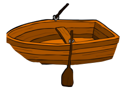
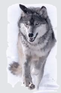
GE The Goose is on the East

CW The Corn is on the West

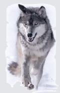
FB The Farmer is in the Boat

This is a safe state.

We will use logic to describe the safe, legal, possible states.

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

We have a dozen propositions.
Each proposition may be true or false.
Each combination of truth values defines a state of the system.

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

These 12 propositions allow 4096 = 2^{12} states.

Some of these are impossible - each thing can only be in one place at a time.
There only 81 possible states. How do we arrive at this number?

How can we use logic to specify the possible states?

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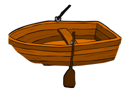
Exactly one of WW WB WE is true

$(WW \oplus WB \oplus WE) \wedge \neg (WW \wedge WB \wedge WE)$

We need a condition like this for each row.

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

Some of the 81 possible states are not legal.
The farmer can only take one load in the boat.

How many of the possible states have at most the farmer and one load in the boat?

How can we use logic to specify the legal states?

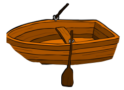
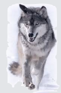
32

At most one load can go in the boat

$\neg (WB \wedge CB) \wedge \neg (WB \wedge GB) \wedge \neg (CB \wedge GB)$

If there is a load in the boat, the farmer must be in the boat

$(WB \vee CB \vee GB) \rightarrow FB$

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

Some of the legal, possible states are not safe.
The farmer cannot safely leave the wolf with the goose or the goose with the corn.

How many of the legal, possible states are safe?

How can we use logic to specify the safe states?

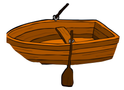
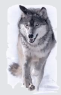
33

If the Wolf and the Goose are together, then the Farmer must be present.

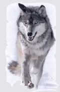
$(WW \wedge GW) \rightarrow FW$

$(WE \wedge GE) \rightarrow FE$

similarly for the Goose and the Corn

	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

Once you have identified the safe, legal, possible states,
you can draw a diagram showing the possible transitions from one state to another.

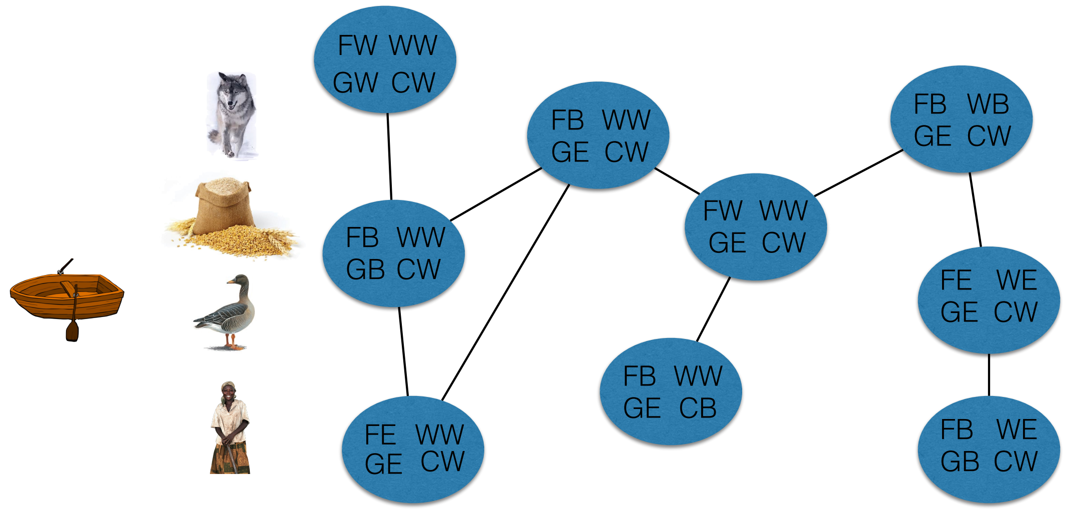
	West		East
	WW	WB	WE
	CW	CB	CE
	GW	GB	GE
	FW	FB	FE

A farmer has to get a wolf, a goose, and a sack of corn across a river.

She has a boat, which can only carry her and one other thing.

If the wolf and the goose are left together, the wolf will eat the goose.

If the goose and the corn are left together, the chicken will eat the corn.



A farmer has to get a wolf, a goose, and a sack of corn across a river.

How can we use logic to specify the transitions?

The

Mathematical Gazette

A JOURNAL OF THE MATHEMATICAL ASSOCIATION

Vol. 73 June 1989 No. 464

The jealous husbands and **The missionaries and cannibals**

IAN PRESSMAN AND DAVID SINGMASTER

The classical river crossing problem of the jealous husbands involves three couples who have to cross a river using a boat that holds just two people. The jealousy of the husbands requires that no wife can be in the presence of another man without her husband being present. This can be accomplished in 11 crossings (i.e. one-way trips). Tartaglia gave a sketchy solution for four couples but Bachet pointed out that this was erroneous and that four couples could not get across the river. In 1879, De Fontenay pointed out that four or more couples could cross the river if there was an island in the river and gave a solution for n couples in $8n - 8$ crossings. Dudeney improved the solution for $n = 4$ and Ball noted that this gives $6n - 7$ crossings for n couples.

From the results of a computer search, we have discovered solutions in 16 crossings for $n = 4$ and in $4n + 1$ crossings for $n > 4$ and we have proven that these are the minimal number of crossings. We have also found that De Fontenay's solution should be in $8n - 6$ crossings and that this is the minimal number of crossings when trips from bank to bank are prohibited.

The more recent missionaries and cannibals problem has n of each type of person and the conditions are that the cannibals must never outnumber the missionaries at any location. This is a proper weakening of the jealous husbands problem. When bank-to-bank crossings are prohibited, De Fontenay's method already uses the least possible number of crossings, even disregarding any conditions, hence is also optimal for this version of the problem. When bank-to-bank crossings are permitted, the 16 crossing solution for the jealous husbands can be reduced to 15 and this generates a solution in $4n - 1$ crossings, which is the minimal number of crossings for $n \geqslant 3$.

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How can we use propositional logic to model the jealous husbands problem?

How many legal safe states are there for this problem?

Can we use propositional logic to model the missionaries and cannibals problem?

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Many problems can be encoded in logic.

A River-Crossing Problem in Cross-Cultural Perspective

MARCIA ASCHER
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1. Introduction Most mathematicians react with interest to the challenge of a logical puzzle. In fact, some story puzzles have become such favorites that many of us cannot even recall where we learned them. Perhaps one of the best known is the puzzle in which a man must ferry across a river a wolf, a goat, and a head of cabbage. The difficulty is that the available boat can only carry him and one other thing but *neither the wolf and goat nor goat and cabbage can be left alone together.* Story puzzles are simple and accessible because they do not rely on any particular body of knowledge and yet they are mathematical in that a stated goal must be achieved under a given set of logical constraints. Attention to logic, as evidenced by the existence of these puzzles, is not the exclusive province of any one culture or subculture. Here, the river-crossing problem, in African cultures as well as in Western culture, will be used as an explicit example of the panhuman concern for mathematical ideas. Story puzzles are expressions of their cultures and so variations will be seen in the characters, the settings and the way in which the logical problem is framed.

2. Western versions The Western origin of the wolf, goat, and cabbage puzzle is most often attributed to a set of 53 problems designed to challenge youthful minds, "Propositiones and acuendos iuvenes." Although circulated around the year 1000, Alcuin of York (735–804) is said to have authored these as he referred to them in a letter to his most famous student, Charlemagne. The solution given by these works is to carry over the goat, then transport the wolf and return with the goat, then carry over the cabbage, then carry over the goat. A second solution, which simply interchanges the wolf and cabbage, is often attributed to the French mathematician Chuquet in 1484 but is found even earlier in the twelfth century in Germany in the succinct form of Latin hexameter [**1**, **4**, **5**, **23**].

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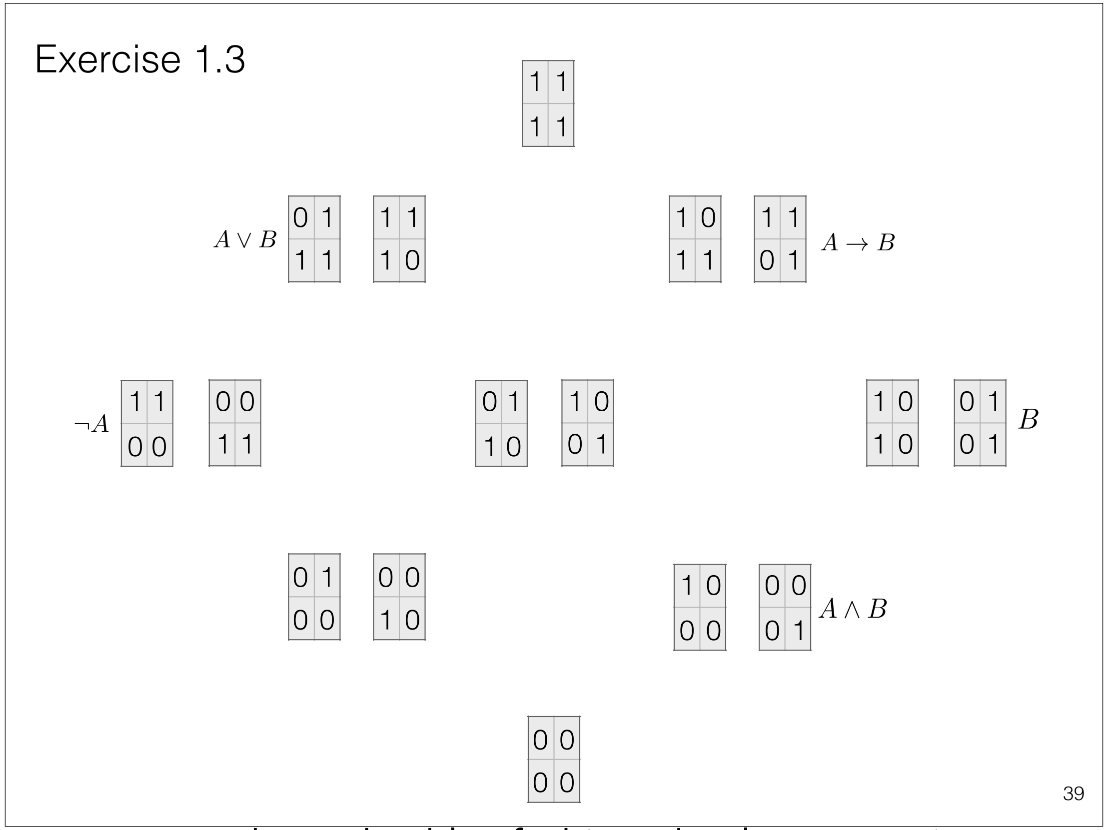
Once you have worked out the set of safe, legal, possible states, and the possible transitions between them, you will have fully analysed this ancient problem - and the conceptual tools you have used can be applied to many different problems.



Each of the 16 2x2 tables above represents the truth table of a binary boolean operation.

Label each table with a boolean expression for which it is the truth table (five tables are already labelled – begin by checking whether these are correct).

How many of the binary operations actually depend on both variables?

How many depend on only one variable?

How many depend on no variables?