Informatics 1

Lecture 6 Satisfiability

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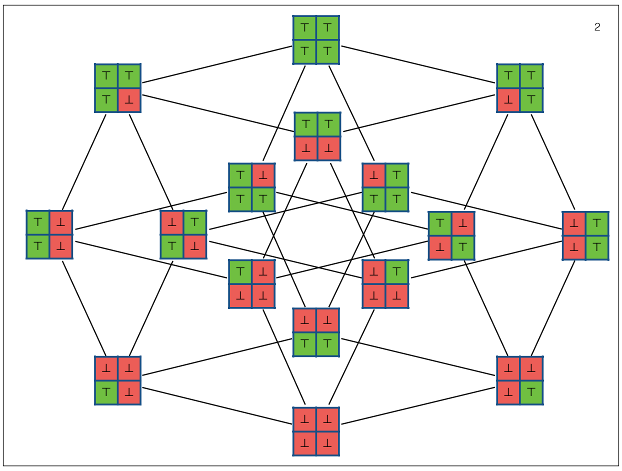
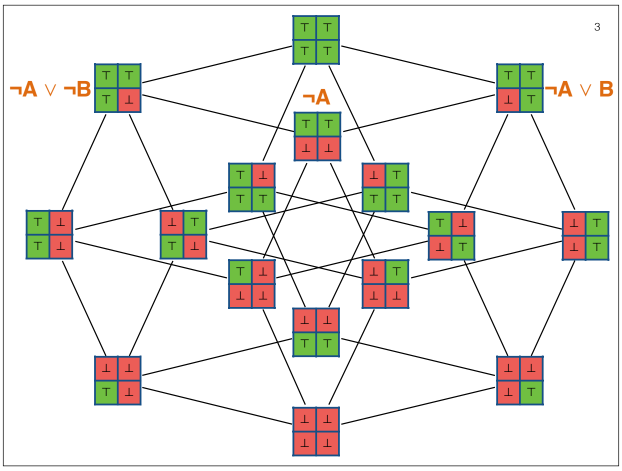
Many practical problems can be phrased as constraint satisfaction problems. For many combinatorial problems the constraints can be expressed in propositional logic.

In this lecture we look at a particularly simple case, known as 2-SAT, where each constraint is a disjunction involving only two literals.



This diagram shows the truth tables for the 16 possible boolean functions of two variables.

We can also view it as a diagram of the subsets of a situation with four individuals, each representative of one of the four possible combinations of two boolean properties A and B. Each boolean function corresponds to a property P, and the diagram shows { x | P(x) }.



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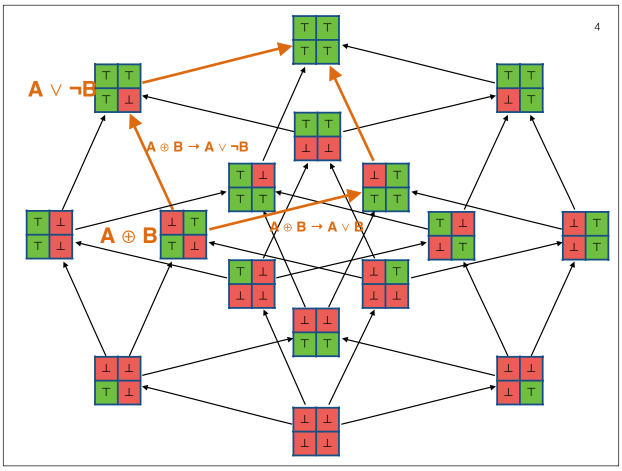
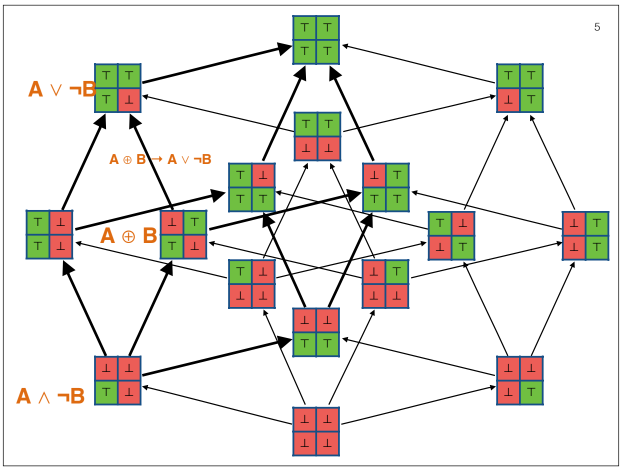


Each line in the diagram represents the addition of an additional element to the set.

Each arrow represents a valid implication



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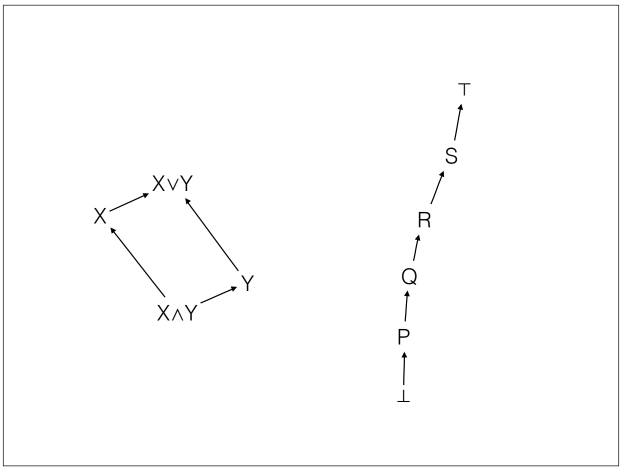


X∨Y
X
Y
X∧Y
⊤
S
R
Q
P
⊥

Ordering

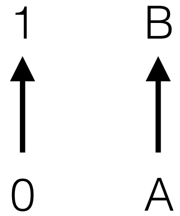
A→B	⊥	⊤
⊥	⊤	⊤
⊤	⊥	⊤

for 0-1 truth values,
$A \to B = \top$ iff
$A \leq B$

if $A \to B = \top$ then
$\{ x \mid A\} \subseteq \{ x \mid B\}$

In any Boolean algebra, we define

$$A \leq B \text{ iff } A \to B = \top \text{ iff } A \wedge B = A \text{ iff } A \vee B = B$$



$0 \leq 1$

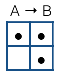
$\bot \leq \top$

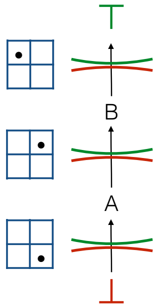
for booleans

$A \rightarrow B = \top$

iff

$A \leq B$





Suppose A → B
there are three possible
truth valuations for A and B
(we exclude only (A = ⊤, B = ⊥))

Propositions are ordered
by $x \leq y$ iff $x \to y = \top$
Any valid truth assignment
must draw a line
between ⊥ and ⊤

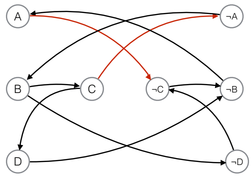
Binary constraints

You may not take both Archeology and Chemistry
If you take Biology you must take Chemistry
You must take Biology or Archeology
If you take Chemistry you must take Divinity
You may not take both Divinity and Biology

$$(\neg A \vee \neg C) \wedge (\neg B \vee C) \wedge (B \vee A) \wedge (\neg C \vee D) \wedge (\neg D \vee \neg B)$$

$$(A \rightarrow \neg C) \wedge (B \rightarrow C) \wedge (\neg B \rightarrow A) \wedge (C \rightarrow D) \wedge (D \rightarrow \neg B)$$

(¬A∨¬C)∧(¬B∨C)∧(B∨A)∧(¬C∨D)∧(¬D∨¬B)



(¬A∨¬C)∧(¬B∨C)∧(B∨A)∧(¬C∨D)∧(¬D∨¬B)

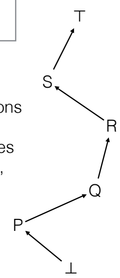
≡

(A→¬C)∧(B→C)∧(¬B→A)∧(C→D)∧(D→¬B)

$P \rightarrow Q$
$\wedge$
$Q \rightarrow R$
$\wedge$
$R \rightarrow S$

$\neg P \vee Q$
$\wedge$
$\neg Q \vee R$
$\wedge$
$\neg R \vee S$

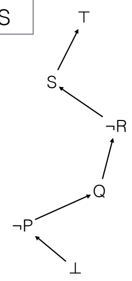
If we have a chain of n-1 implications
between n variables
we can draw the line in n+1 places
making any number, from 0 to n,
of these variables true.



$$\neg P \to Q \\ \wedge \\ Q \to \neg R \\ \wedge \\ \neg R \to S$$

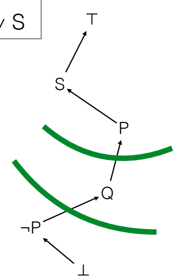
$$P \vee Q \\ \wedge \\ \neg Q \vee \neg R \\ \wedge \\ R \vee S$$

If some of the variables are negated we can do the same (but making the negated variables false when they fall above the line and true when they fall below)



$$\neg P \to Q$$
$$\wedge$$
$$Q \to P$$
$$\wedge$$
$$P \to S$$

$$P \vee Q$$
$$\wedge$$
$$\neg Q \vee P$$
$$\wedge$$
$$\neg P \vee S$$



If a variable appears together with its negation, we have to draw the line between them.

Here, P must be true.

$(\neg P \to P) \to P$
is a tautology

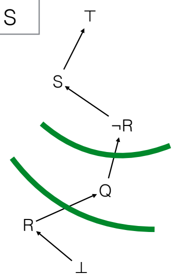
$R \rightarrow Q$
$\wedge$
$Q \rightarrow \neg R$
$\wedge$
$\neg R \rightarrow S$

$\neg R \vee Q$
$\wedge$
$\neg Q \vee \neg R$
$\wedge$
$R \vee S$

If a variable appears together with its negation, we have to draw the line between them.

Here, R must be false.

$(R \rightarrow \neg R) \rightarrow \neg R$
is a tautology

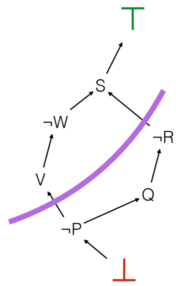


$$\begin{array}{c} \neg P \to V \\ \wedge \\ V \to \neg W \\ \wedge \\ \neg W \to S \end{array} \;\bigwedge\; \begin{array}{c} \neg P \to Q \\ \wedge \\ Q \to \neg R \\ \wedge \\ \neg R \to S \end{array}$$

$$\begin{array}{c} P \vee V \\ \wedge \\ \neg V \vee \neg W \\ \wedge \\ W \vee S \end{array} \;\bigwedge\; \begin{array}{c} P \vee Q \\ \wedge \\ \neg Q \vee \neg R \\ \wedge \\ R \vee S \end{array}$$

The same trick works if our implications form a partial order.
But we have more options since we can draw a wavy line.

The **arrow rule** says that, whenever our line cuts an arrow, then the head must be on the side of true and the tail on the side of false.



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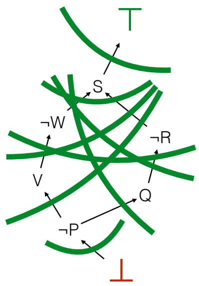
Not all of the valid truth assignments are represented in this diagram.

How many are missing?



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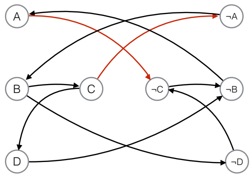
(¬A∨¬C)∧(¬B∨C)∧(B∨A)∧(¬C∨D)∧(¬D∨¬B)

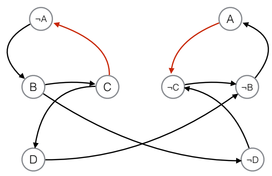


(¬A∨¬C)∧(¬B∨C)∧(B∨A)∧(¬C∨D)∧(¬D∨¬B)

≡

(A→¬C)∧(B→C)∧(¬B→A)∧(C→D)∧(D→¬B)

(¬A∨¬C)∧(¬B∨C)∧(B∨A)∧(¬C∨D)∧(¬D∨¬B)



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≡

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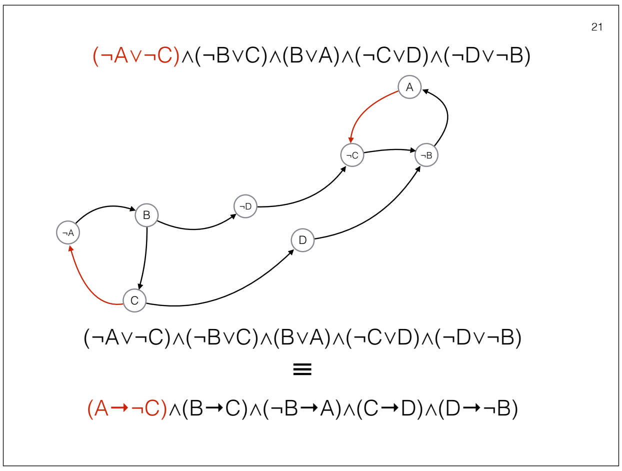
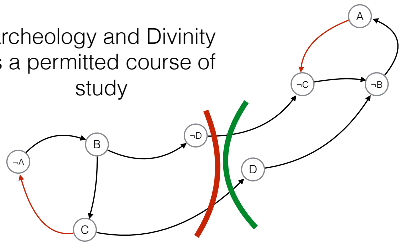


If we have cycles of implications, then all nodes in the cycle must take the same truth value.

(¬A∨¬C)∧(¬B∨C)∧(B∨A)∧(¬C∨D)∧(¬D∨¬B)

Archeology and Divinity is a permitted course of study



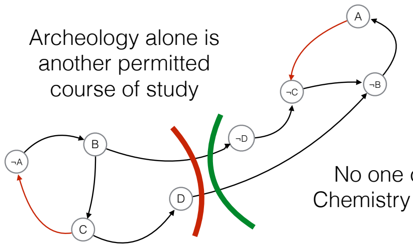
(¬A∨¬C)∧(¬B∨C)∧(B∨A)∧(¬C∨D)∧(¬D∨¬B)

≡

(A→¬C)∧(B→C)∧(¬B→A)∧(C→D)∧(D→¬B)

$(\neg A \lor \neg C) \land (\neg B \lor C) \land (B \lor A) \land (\neg C \lor D) \land (\neg D \lor \neg B)$

Archeology alone is another permitted course of study

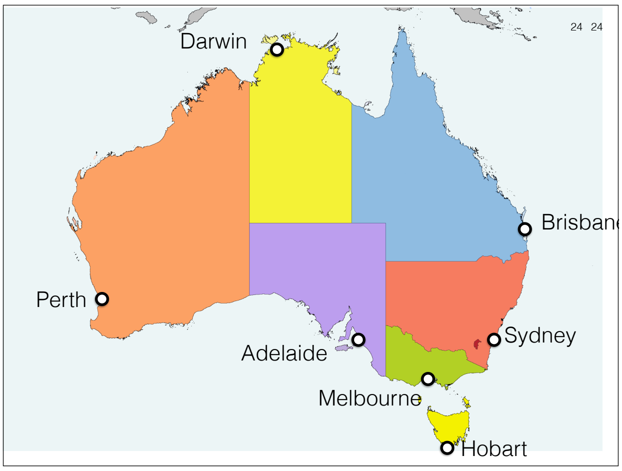


No one can take Chemistry or Biology

$(\neg A \lor \neg C) \land (\neg B \lor C) \land (B \lor A) \land (\neg C \lor D) \land (\neg D \lor \neg B)$

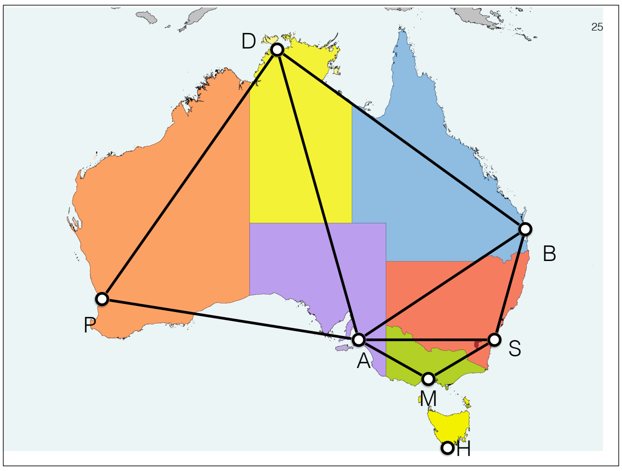
$$\equiv$$

$(A \rightarrow \neg C) \land (B \rightarrow C) \land (\neg B \rightarrow A) \land (C \rightarrow D) \land (D \rightarrow \neg B)$



This map uses four colours and colours two adjacent states with the same colour.

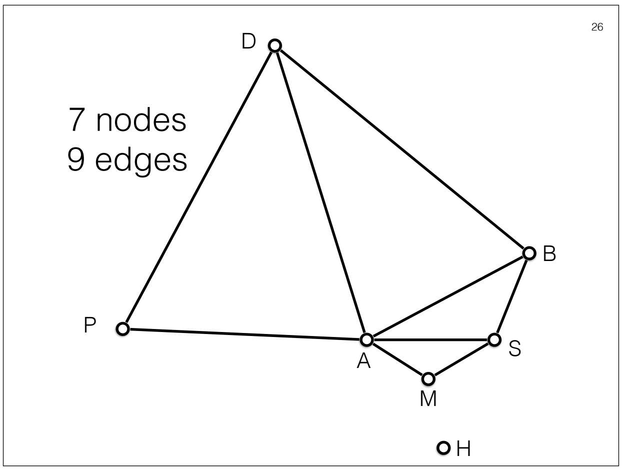
Can we use 3 colours to colour the map so that no two adjacent regions have the same colour



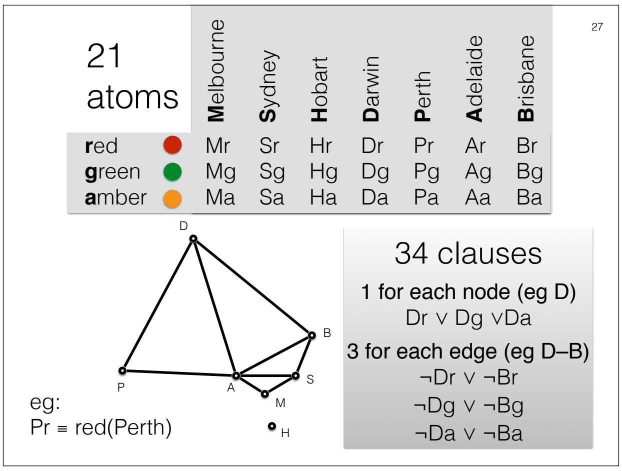
Add a node for each region (we place it at the capital city).

The constraints are represented by a graph - a (symmetric) binary relation.

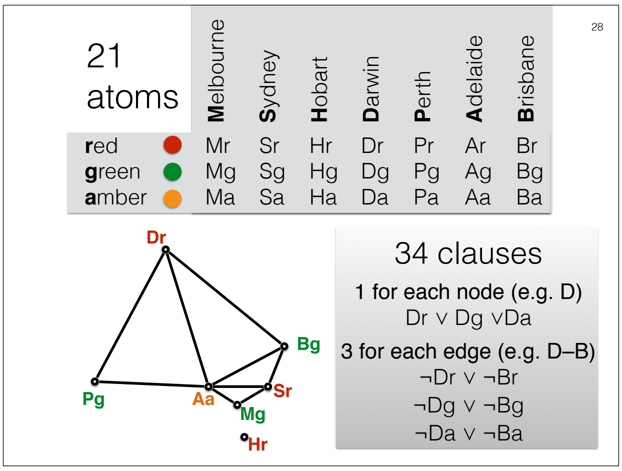
We link two capitals if their states share a common border.



We can get rid of the map and focus on the graph.



We introduce atomic propositions Pr = red(Perth), and express the constraints



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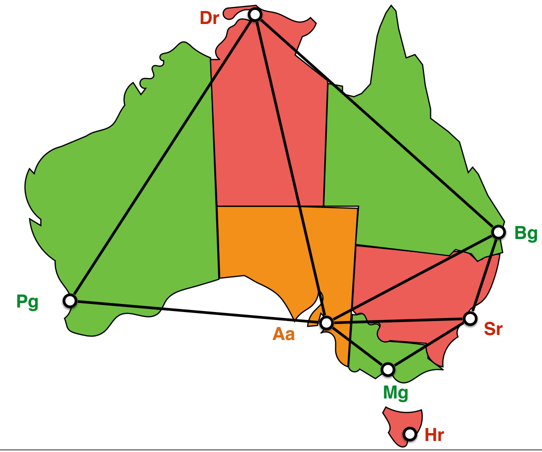
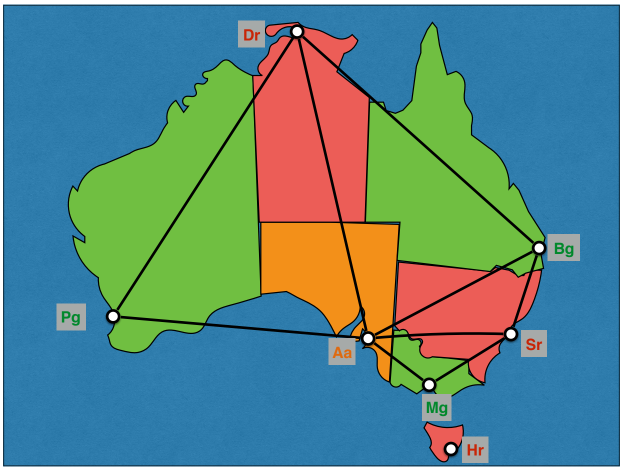


Dr
Bg
Pg
Aa
Sr
Mg
Hr



To include the sea we need a fourth colour.