Informatics 1

Computation and Logic

Boolean Algebra CNF DNF

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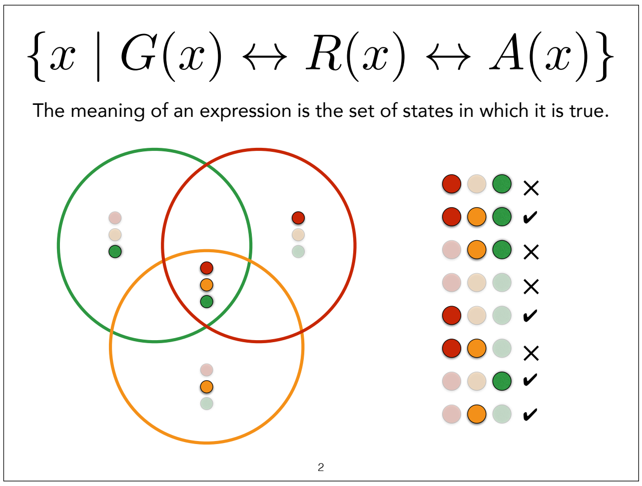
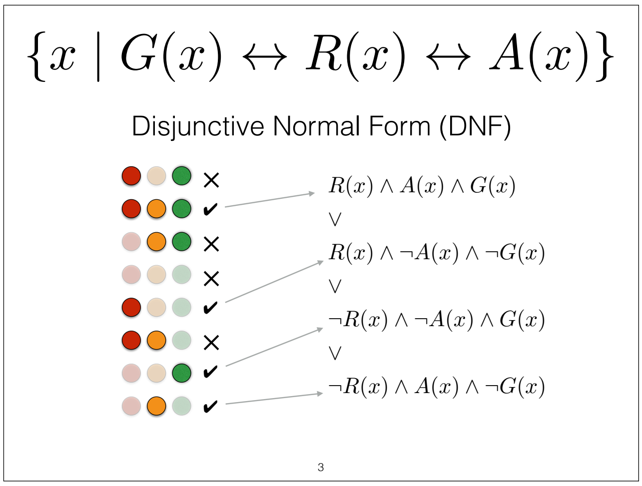


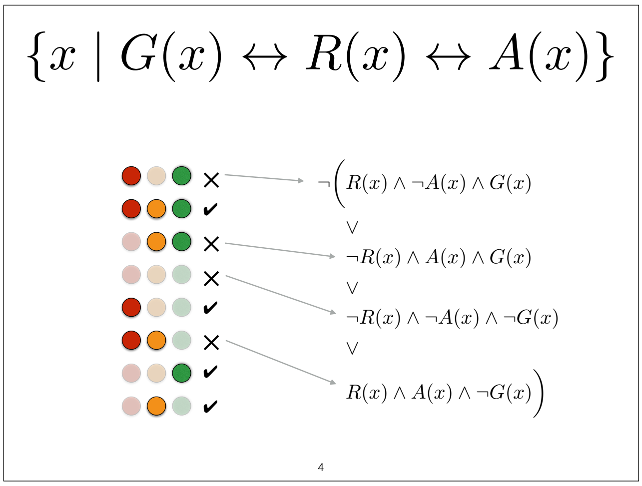
To determine whether to expressions are equivalent, we can check whether they give the same values for all 2^n states of the system.

The meaning of an expression is the set of states in which it is true.

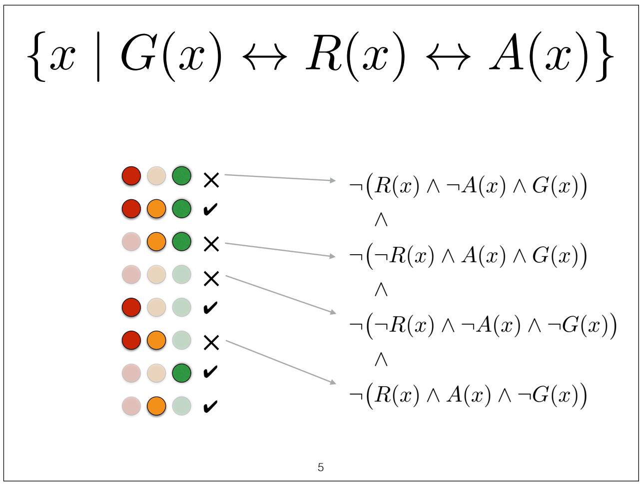


A boolean function of three variables is given by a truth table with eight entries

We can easily write down a disjunction of terms, each one of which corresponds to a single state in which the function is true. This is called a Disjunctive Normal Form (DNF)



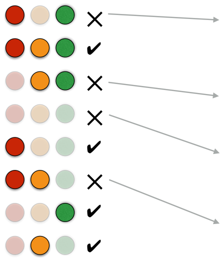
We can do things differently. Here we say that we are not in a state where the function is false



Using de Morgan, this becomes a conjunction of negated conjunctions.

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$

Conjunctive Normal Form (CNF)



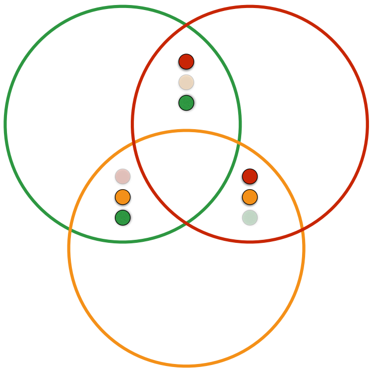
$$(\neg R(x) \vee A(x) \vee \neg G(x))$$
$$\wedge$$
$$(R(x) \vee \neg A(x) \vee \neg G(x))$$
$$\wedge$$
$$(R(x) \vee A(x) \vee G(x))$$
$$\wedge$$
$$(\neg R(x) \vee \neg A(x) \vee G(x))$$

Using de Morgan again, this becomes a conjunction of disjunctions.

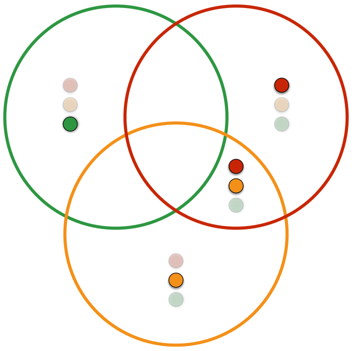
We will return to CNF later.

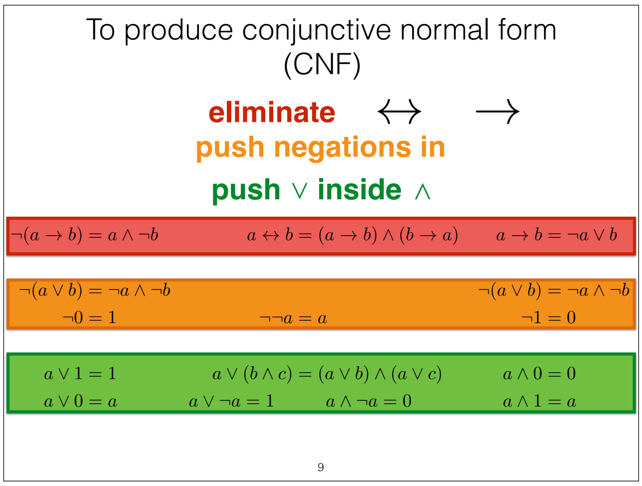
CNF

Exercise 2.2 Generate CNF for this subset

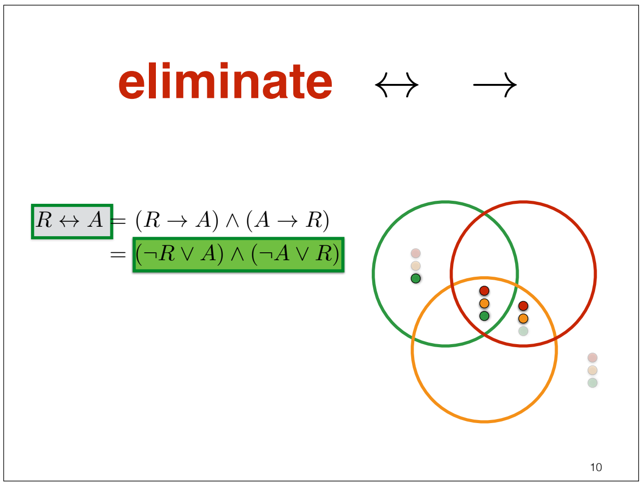


Exercise 2.3 Generate CNF for this subset





We can transform any Boolean expression algebraically to create an equivalent CNF



In this case, once we have eliminated implications, we have CNF.
Clauses with only two literals correspond to implications.

eliminate $\leftrightarrow$ $\rightarrow$

$$R \leftrightarrow A = (R \rightarrow A) \wedge (A \rightarrow R)$$
$$= (\neg R \vee A) \wedge (\neg A \vee R)$$

$$G \leftrightarrow (R \leftrightarrow A)$$
$$= (\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R)))$$
$$\wedge$$
$$(\neg((\neg R \vee A) \wedge (\neg A \vee R)) \vee G)$$

Here, we use the previous result to re-write the part in parentheses.

push negations in

$$
\begin{aligned}
G \leftrightarrow (R \leftrightarrow A) \\
&= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big) \\
&\quad \wedge \\
&\quad \big(\neg((\neg R \vee A) \wedge (\neg A \vee R)) \vee G\big) \\
&= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big) \\
&\quad \wedge \\
&\quad \big((\neg(\neg R \vee A) \vee \neg(\neg A \vee R)) \vee G\big)
\end{aligned}
$$

push negations in

$$G \leftrightarrow (R \leftrightarrow A)$$

$$= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big)$$
$$\wedge$$
$$\big(\neg((\neg R \vee A) \wedge (\neg A \vee R)) \vee G\big)$$

$$= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big)$$
$$\wedge$$
$$\big((\neg(\neg R \vee A) \vee \neg(\neg A \vee R)) \vee G\big)$$

$$= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big)$$
$$\wedge$$
$$\big(((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G\big)$$

push ∨ inside ∧

$$
\begin{aligned}
G \leftrightarrow (R \leftrightarrow A) \\
&= \big(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))\big) \\
&\quad \wedge \\
&\quad \big(((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G\big) \\
&= \big(((\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R))\big) \\
&\quad \wedge \\
&\quad \big(((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G\big)
\end{aligned}
$$

push ∨ inside ∧

$$\begin{aligned} G \leftrightarrow (R \leftrightarrow A) \\ &= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\ &\quad \wedge \\ &\quad \big(((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G\big) \\ &= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\ &\quad \wedge \\ &\quad \big(((R \vee A) \wedge (\neg A \vee A) \wedge (R \vee \neg R) \wedge (\neg A \vee \neg R)) \vee G\big) \end{aligned}$$

simplify

$$\neg A \lor A = \top$$
$$R \lor \neg R = \top$$
$$x \land \top = x$$

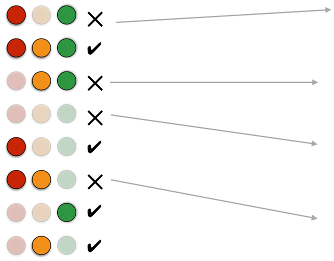
$$\begin{aligned}
G \leftrightarrow (R \leftrightarrow A) \\
&= (\neg G \lor \neg R \lor A) \land (\neg G \lor \neg A \lor R) \\
&\quad \land \\
&\quad \big(((R \lor A) \land (\neg A \lor A) \land (R \lor \neg R) \land (\neg A \lor \neg R)) \lor G\big) \\
&= (\neg G \lor \neg R \lor A) \land (\neg G \lor \neg A \lor R) \\
&\quad \land \\
&\quad \big(((R \lor A) \land (\neg A \lor \neg R)) \lor G\big)
\end{aligned}$$

push $\vee$ inside $\wedge$

$$
\begin{aligned}
G \leftrightarrow (R \leftrightarrow A) \\
&= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\
&\quad \wedge \\
&\quad (((R \vee A) \wedge (\neg A \vee \neg R)) \vee G) \\
&= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\
&\quad \wedge \\
&\quad (R \vee A \vee G) \wedge (\neg A \vee \neg R \vee G)
\end{aligned}
$$

check!

$$G \leftrightarrow (R \leftrightarrow A) =$$



$$(\neg G \vee \neg R \vee A)$$
$$\wedge$$
$$(\neg G \vee \neg A \vee R)$$
$$\wedge$$
$$(R \vee A \vee G)$$
$$\wedge$$
$$(\neg A \vee \neg R \vee G)$$

(A?B:C)

if A then B else C