

Informatics 1

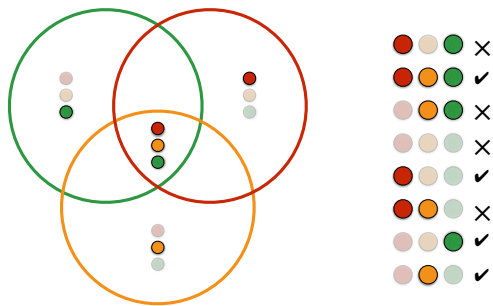
Computation and Logic
Boolean Algebra CNF DNF

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1

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$

The meaning of an expression is the set of states in which it is true.



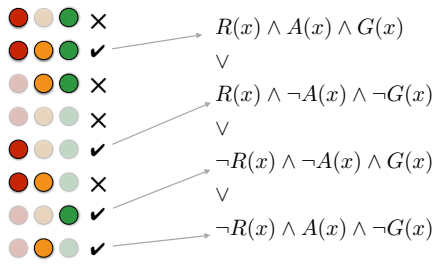
2

To determine whether two expressions are equivalent, we can check whether they give the same values for all 2^n states of the system.

The meaning of an expression is the set of states in which it is true.

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$

Disjunctive Normal Form (DNF)

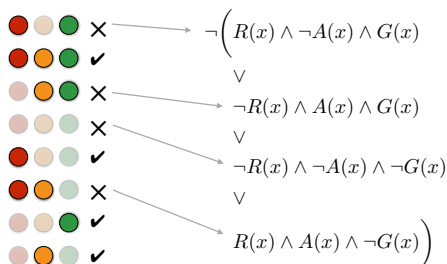


3

A boolean function of three variables is given by a truth table with eight entries

We can easily write down a disjunction of terms, each one of which corresponds to a single state in which the function is true. This is called a Disjunctive Normal Form (DNF)

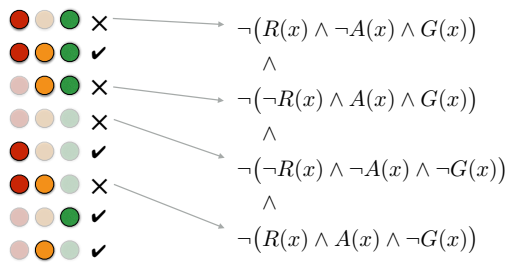
$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$



4

We can do things differently. Here we say that we are not in a state where the function is false

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$

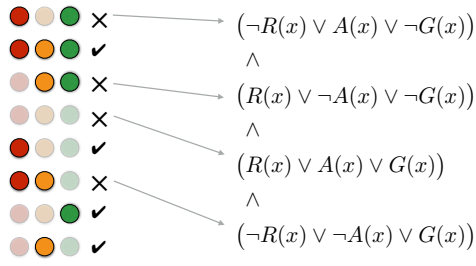


5

Using de Morgan, this becomes a conjunction of negated conjunctions.

$$\{x \mid G(x) \leftrightarrow R(x) \leftrightarrow A(x)\}$$

Conjunctive Normal Form (CNF)



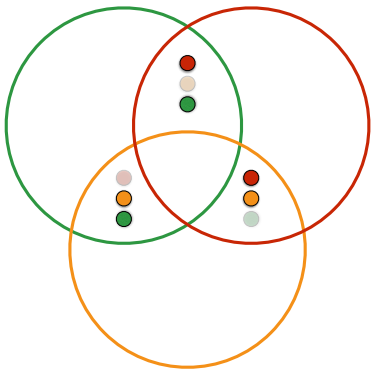
6

Using de Morgan again, this becomes a conjunction of disjunctions.

We will return to CNF later.

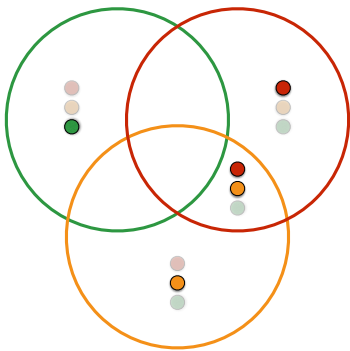
CNF

Exercise 2.2 Generate CNF for this subset



7

Exercise 2.3 Generate CNF for this subset



8

To produce conjunctive normal form (CNF)

eliminate \leftrightarrow \rightarrow
push negations in
push \vee inside \wedge

$$\neg(a \rightarrow b) = a \wedge \neg b \quad a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a) \quad a \rightarrow b = \neg a \vee b$$

$$\neg(a \vee b) = \neg a \wedge \neg b \quad \neg(a \vee b) = \neg a \wedge \neg b$$

$$\neg 0 = 1 \quad \neg \neg a = a \quad \neg 1 = 0$$

$$a \vee 1 = 1 \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad a \wedge 0 = 0$$

$$a \vee 0 = a \quad a \vee \neg a = 1 \quad a \wedge \neg a = 0 \quad a \wedge 1 = a$$

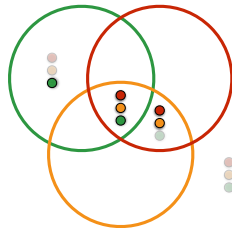
9

We can transform any Boolean expression algebraically to create an equivalent CNF

eliminate \leftrightarrow \rightarrow

$$R \leftrightarrow A = (R \rightarrow A) \wedge (A \rightarrow R)$$

$$= (\neg R \vee A) \wedge (\neg A \vee R)$$



10

In this case, once we have eliminated implications, we have CNF.

Clauses with only two literals correspond to implications.

eliminate \leftrightarrow \rightarrow

$$\begin{aligned} R \leftrightarrow A &= (R \rightarrow A) \wedge (A \rightarrow R) \\ &= (\neg R \vee A) \wedge (\neg A \vee R) \end{aligned}$$

$$\begin{aligned} G \leftrightarrow (R \leftrightarrow A) & \\ &= (\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))) \\ &\quad \wedge \\ &\quad (\neg((\neg R \vee A) \wedge (\neg A \vee R)) \vee G) \end{aligned}$$

11

Here, we use the previous result to re-write the part in parentheses.

push negations in

$$\begin{aligned} G \leftrightarrow (R \leftrightarrow A) & \\ &= (\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))) \\ &\quad \wedge \\ &\quad (\neg((\neg R \vee A) \wedge (\neg A \vee R)) \vee G) \\ &= (\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))) \\ &\quad \wedge \\ &\quad ((\neg(\neg R \vee A) \vee \neg(\neg A \vee R)) \vee G) \end{aligned}$$

12

push negations in

$$\begin{aligned}G &\leftrightarrow (R \leftrightarrow A) \\&= (\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))) \\&\quad \wedge \\&\quad (\neg((\neg R \vee A) \wedge (\neg A \vee R)) \vee G) \\&= (\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))) \\&\quad \wedge \\&\quad \boxed{((\neg(\neg R \vee A) \vee \neg(\neg A \vee R)) \vee G)} \\&= (\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R))) \\&\quad \wedge \\&\quad \boxed{((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G}\end{aligned}$$

13

push \vee inside \wedge

$$\begin{aligned}G &\leftrightarrow (R \leftrightarrow A) \\&= \boxed{(\neg G \vee ((\neg R \vee A) \wedge (\neg A \vee R)))} \\&\quad \wedge \\&\quad (((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G) \\&= \boxed{(((\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R))} \\&\quad \wedge \\&\quad (((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G)\end{aligned}$$

14

push \vee inside \wedge

$$\begin{aligned} G &\leftrightarrow (R \leftrightarrow A) \\ &= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\ &\quad \wedge \\ &\quad \left(((R \wedge \neg A) \vee (A \wedge \neg R)) \vee G \right) \\ &= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\ &\quad \wedge \\ &\quad \left((R \vee A) \wedge (\neg A \vee A) \wedge (R \vee \neg R) \wedge (\neg A \vee \neg R) \right) \vee G \end{aligned}$$

15

simplify

$$\begin{aligned} \neg A \vee A &= \top \\ R \vee \neg R &= \top \\ x \wedge \top &= x \end{aligned}$$

$$\begin{aligned} G &\leftrightarrow (R \leftrightarrow A) \\ &= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\ &\quad \wedge \\ &\quad \left((R \vee A) \wedge (\neg A \vee A) \wedge (R \vee \neg R) \wedge (\neg A \vee \neg R) \right) \vee G \\ &= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\ &\quad \wedge \\ &\quad \left((R \vee A) \wedge (\neg A \vee \neg R) \right) \vee G \end{aligned}$$

16

push \vee inside \wedge

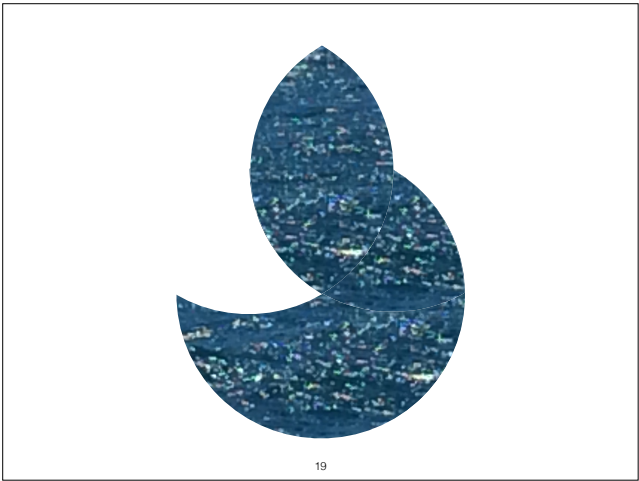
$$\begin{aligned}
 G \leftrightarrow (R \leftrightarrow A) & \\
 &= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\
 &\quad \wedge \\
 &\quad \boxed{((R \vee A) \wedge (\neg A \vee \neg R)) \vee G} \\
 &= (\neg G \vee \neg R \vee A) \wedge (\neg G \vee \neg A \vee R) \\
 &\quad \wedge \\
 &\quad \boxed{(R \vee A \vee G) \wedge (\neg A \vee \neg R \vee G)}
 \end{aligned}$$

17

check!

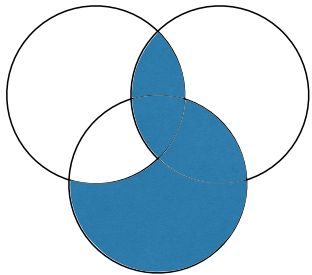
<div style="display: flex; gap: 5px;"> ● ● ● ✗ </div>	\longrightarrow	$(\neg G \vee \neg R \vee A)$
<div style="display: flex; gap: 5px;"> ● ● ● ✓ </div>	\longrightarrow	\wedge
<div style="display: flex; gap: 5px;"> ● ● ● ✗ </div>	\longrightarrow	$(\neg G \vee \neg A \vee R)$
<div style="display: flex; gap: 5px;"> ● ● ● ✗ </div>	\longrightarrow	\wedge
<div style="display: flex; gap: 5px;"> ● ● ● ✓ </div>	\longrightarrow	$(R \vee A \vee G)$
<div style="display: flex; gap: 5px;"> ● ● ● ✗ </div>	\longrightarrow	\wedge
<div style="display: flex; gap: 5px;"> ● ● ● ✓ </div>	\longrightarrow	$(\neg A \vee \neg R \vee G)$
<div style="display: flex; gap: 5px;"> ● ● ● ✓ </div>		

18



19

$(A?B:C)$
if A then B else C



20