Informatics 1

Computation and Logic

Boolean Algebra

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Basic Boolean operations

Boole (1815 – 1864)

$\mathbf{1}, \top$	true, top
$\vee$	disjunction, or
$\wedge$	conjunction, and
$\neg$	negation, not
$\mathbf{0}, \bot$	false, bottom

$$\mathbb{Z}_2 = \{0, 1\}$$

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

−	
0	0
1	1

$$x \wedge y \equiv xy$$
$$x \vee y \equiv x + y - xy$$
$$\neg x \equiv 1 - x$$

Here, we use arithmetic mod 2
The same equations work if we use ordinary arithmetic!

∨	0	1
0	0	1
1	1	1

∧	0	1
0	0	0
1	0	1

¬	
0	1
1	0

If we use one bit (binary unit, 0 or 1) to code each truth value, with $0 \sim \bot$ and $1 \sim \top$

then we can express the logical operations algebraically.

It doesn't matter whether we interpret these expressions in Z or in Z_2

The algebra of sets

$$\mathcal{P}(S) = \{X \mid X \subseteq S\}$$

$X \vee Y = X \cup Y$	union
$X \wedge Y = X \cap Y$	intersection
$\neg X = S \setminus Y$	complement
$0 = \emptyset$	empty set
$1 = S$	entire set

Derived Operations

Definitions:

$$
\begin{aligned}
x \to y &\equiv \neg x \vee y && \text{implication} \\
x \leftarrow y &\equiv x \vee \neg y && \\
x \leftrightarrow y &\equiv (\neg x \wedge \neg y) \vee (x \wedge y) && \text{equality (iff)} \\
x \oplus y &\equiv (\neg x \wedge y) \vee (x \wedge \neg y) && \text{inequality (xor)}
\end{aligned}
$$

Some equations:

$$
\begin{aligned}
x \leftrightarrow y &= (x \to y) \wedge (x \leftarrow y) \\
x \oplus y &= \neg(x \leftrightarrow y) \\
x \oplus y &= \neg x \oplus \neg y \\
x \leftrightarrow y &= \neg(x \oplus y) \\
x \leftrightarrow y &= \neg x \leftrightarrow \neg y
\end{aligned}
$$

an algebraic proof

$$\begin{aligned}(x \leftrightarrow y) \leftrightarrow z &= \neg(x \leftrightarrow y) \leftrightarrow \neg z && (a \leftrightarrow b = \neg a \leftrightarrow \neg b)\\ &= (x \oplus y) \leftrightarrow \neg z && (\neg(a \leftrightarrow b) = a \oplus b)\\ &= (x \oplus y) \oplus z && (a \leftrightarrow \neg b = a \oplus b)\end{aligned}$$

Once we know the rules for iff and xor shown on the right, we can give an algebraic proof that the xor combination of three variables is the same as their iff combination

Boolean connectives

Some equalities:

$$x \vee y = \neg(\neg x \wedge \neg y) \qquad x \wedge y = \neg(\neg x \vee \neg y)$$
$$\neg x = x \to 0 \qquad x \vee y = \neg x \to y$$

We will see that $\wedge$, $\vee$, $\neg$ and $\perp$ are sufficient to define any boolean function. These equations show that $\{\wedge, \neg, \perp\}$, $\{\vee, \neg, \perp\}$, and $\{\to, \perp\}$ are all sufficient sets.

Boolean Algebra

$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	associative
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	distributive
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	commutative
$x \vee 0 = x$	$x \wedge 1 = x$	identity
$x \vee 1 = 1$	$x \wedge 0 = 0$	annihilation
$x \vee x = x$	$x \wedge x = x$	idempotent
$x \vee \neg x = 1$	$\neg x \wedge x = 0$	complements

$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$	absorbtion
$\neg(x \vee y) = \neg x \wedge \neg y$	$\neg(x \wedge y) = \neg x \vee \neg y$	de Morgan
$\neg\neg x = x$	$x \to y = \neg x \leftarrow \neg y$	

The equations above the gap define a Boolean algebra.

Those below the line follow from these.

Exercise 2.1

Which of the following rules are *not* valid for arithmetic?
Which of the rules are *not* valid for arithmetic in Z_2?

$x + (y + z) = (x + y) + z$	$x \times (y \times z) = (x \times y) \times z$	associative
$x + (y \times z) = (x + y) \times (x + z)$	$x \times (y + z) = (x \times y) + (x \times z)$	distributive
$x + y = y + x$	$x \times y = y \times x$	commutative
$x + 0 = x$	$x \times 1 = x$	identity
$x + 1 = 1$	$x \times 0 = x$	annihilation
$x + x = x$	$x \times x = x$	idempotent
$x + (x \times y) = x$	$x + (x \times y) = x$	absorbtion
$x + -x = 1$	$x \times -x = 0$	complements

Exercise 2.4 (for mathematicians)

In any Boolean algebra, define,

$$x \leq y \equiv x \wedge y = x$$

1. Show that, for any x, y, and z,

$$\begin{gathered} 0 \leq x \text{ and } x \leq x \text{ and } x \leq 1 \\ x \rightarrow y = \top \text{ iff } x \leq y \\ \text{if } x \leq y \text{ and } y \leq z \text{ then } x \leq z \\ \text{if } x \leq y \text{ and } y \leq x \text{ then } x = y \\ \text{if } x \leq y \text{ then } \neg y \leq \neg x \end{gathered}$$

2. Show that, in any Boolean Algebra,

$$x \wedge y = x \text{ iff } x \vee y = y$$