Informatics 1

Computation and Logic

Boolean Algebra

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Basic Boolean operations

Boole (1815 – 1864)

$\mathbf{1}, \top$	true, top
$\vee$	disjunction, or
$\wedge$	conjunction, and
$\neg$	negation, not
$\mathbf{0}, \bot$	false, bottom

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$\mathbb{Z}_2 = \{0, 1\}$

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

−	
0	0
1	1

$$x \wedge y \equiv xy$$
$$x \vee y \equiv x + y - xy$$
$$\neg x \equiv 1 - x$$

Here, we use arithmetic mod 2
The same equations work if we use ordinary arithmetic!

∨	0	1
0	0	1
1	1	1

∧	0	1
0	0	0
1	0	1

¬	
0	1
1	0

The algebra of sets

$$\mathcal{P}(S) = \{X \mid X \subseteq S\}$$

$X \vee Y = X \cup Y$	union
$X \wedge Y = X \cap Y$	intersection
$\neg X = S \setminus Y$	complement
$0 = \emptyset$	empty set
$1 = S$	entire set

Derived Operations

Definitions:

$$x \to y \equiv \neg x \lor y \qquad \text{implication}$$
$$x \leftarrow y \equiv x \lor \neg y$$
$$x \leftrightarrow y \equiv (\neg x \land \neg y) \lor (x \land y) \qquad \text{equality (iff)}$$
$$x \oplus y \equiv (\neg x \land y) \lor (x \land \neg y) \qquad \text{inequality (xor)}$$

Some equations:

$$x \leftrightarrow y = (x \to y) \land (x \leftarrow y)$$
$$x \oplus y = \neg(x \leftrightarrow y)$$
$$x \oplus y = \neg x \oplus \neg y$$
$$x \leftrightarrow y = \neg(x \oplus y)$$
$$x \leftrightarrow y = \neg x \leftrightarrow \neg y$$

an algebraic proof

$$(x \leftrightarrow y) \leftrightarrow z = \neg(x \leftrightarrow y) \leftrightarrow \neg z \qquad (a \leftrightarrow b = \neg a \leftrightarrow \neg b)$$
$$= (x \oplus y) \leftrightarrow \neg z \qquad (\neg(a \leftrightarrow b) = a \oplus b)$$
$$= (x \oplus y) \oplus z \qquad (a \leftrightarrow \neg b = a \oplus b)$$

Boolean connectives

Some equalities:

$$x \vee y = \neg(\neg x \wedge \neg y) \qquad x \wedge y = \neg(\neg x \vee \neg y)$$
$$\neg x = x \rightarrow 0 \qquad x \vee y = \neg x \rightarrow y$$

We will see that $\wedge$, $\vee$, $\neg$ and $\bot$ are sufficient to define any boolean function. These equations show that $\{\wedge, \neg, \bot\}$, $\{\vee, \neg, \bot\}$, and $\{\rightarrow, \bot\}$ are all sufficient sets.

Boolean Algebra

$x \vee (y \vee z) = (x \vee y) \vee z$	$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	associative
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	distributive
$x \vee y = y \vee x$	$x \wedge y = y \wedge x$	commutative
$x \vee 0 = x$	$x \wedge 1 = x$	identity
$x \vee 1 = 1$	$x \wedge 0 = 0$	annihilation
$x \vee x = x$	$x \wedge x = x$	idempotent
$x \vee \neg x = 1$	$\neg x \wedge x = 0$	complements
$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$	absorbtion
$\neg(x \vee y) = \neg x \wedge \neg y$	$\neg(x \wedge y) = \neg x \vee \neg y$	de Morgan
$\neg\neg x = x$	$x \rightarrow y = \neg x \leftarrow \neg y$	

Exercise 2.1

Which of the following rules are *not* valid for arithmetic?
Which of the rules are *not* valid for arithmetic in Z_2?

$x + (y + z) = (x + y) + z$	$x \times (y \times z) = (x \times y) \times z$	associative
$x + (y \times z) = (x + y) \times (x + z)$	$x \times (y + z) = (x \times y) + (x \times z)$	distributive
$x + y = y + x$	$x \times y = y \times x$	commutative
$x + 0 = x$	$x \times 1 = x$	identity
$x + 1 = 1$	$x \times 0 = x$	annihilation
$x + x = x$	$x \times x = x$	idempotent
$x + (x \times y) = x$	$x + (x \times y) = x$	absorbtion
$x + -x = 1$	$x \times -x = 0$	complements

Exercise 2.4 (for mathematicians)

In any Boolean algebra, define,

$$x \leq y \equiv x \wedge y = x$$

1. Show that, for any x, y, and z,

$$0 \leq x \text{ and } x \leq x \text{ and } x \leq 1$$
$$x \to y = \top \;\text{ iff } x \leq y$$
$$\text{if } x \leq y \text{ and } y \leq z \text{ then } x \leq z$$
$$\text{if } x \leq y \text{ and } y \leq x \text{ then } x = y$$
$$\text{if } x \leq y \text{ then } \neg y \leq \neg x$$

2. Show that, in any Boolean Algebra,

$$x \wedge y = x \text{ iff } x \vee y = y$$