# Negation

In this lecture we extend the scope of our proofs to deal with negation.

# Problem: A Medieval Logical Puzzle

- From “Codex Paradoxicus”: “Could it be the case that seven angels can sit on the head of a pin, and that it is not the case both that Jorge of Burgos wrote “The Name of the Rose” and that Astana is the capital of Uqbar, and that Uqbar’s capital is Astana or that seven angels can’t sit on the head of a pin, and that it is not the case that Jorge of Burgos wrote “The Name of the Rose”?”

# Proof Rules for Negation

Added to those we saw in the last lecture

| Proof | Sub-proofs | |
|---|---|---|
| $F \vdash$ not(A) | [ A\|$F$ ] $\vdash$ false | |
| $F \vdash$ B | not(A) ∈ $F$ | $F \vdash$ A |
| $F \vdash$ A | $F \vdash$ not(not(A)) | |

The purpose of these rules is to provide positive evidence that an expression is false.

# A Proof Tree

[A, not(A and B)] ⊢ not(B)

[B,A, not(A and B)] ⊢ false

not(A and B) ∈ [B,A, not(A and B)]

[B,A, not(A and B)] ⊢ A and B

[B,A, not(A and B)] ⊢ A

[B,A, not(A and B)] ⊢ B

A ∈ [B,A, not(A and B)]

B ∈ [B,A, not(A and B)]

# Automatic Verifiability of Proofs

- Checking whether a proof rule has been applied correctly can be automated

# A Proof Tree

[A, not(A and B)] ⊢ not(B)

[B,A, not(A and B)] ⊢ false

not(A and B) ∈ [B,A, not(A and B)]

[B,A, not(A and B)] ⊢ A and B

[B,A, not(A and B)] ⊢ A

[B,A, not(A and B)] ⊢ B

A ∈ [B,A, not(A and B)]

B ∈ [B,A, not(A and B)]

# Automatic Verifiability of Proofs

- Checking whether a proof rule has been applied correctly can be automated
- However, it is not obvious how to *generate* a proof of a valid sequent

# Non-determinism

- Fundamental concept in computer science
- Intuitively, non-determinism refers to freedom of making guesses
- When constructing a proof, at each step one can *guess* which proof rule to apply next
- Some sequences of guesses are good, i.e., lead to a correct proof quickly, while others are bad
- We will later see how to formalize this concept for computation

# Problem: A Medieval Logical Puzzle

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p = “Seven angels can sit on the head of a pin”
q = “Astana is the capital of Uqbar”
r = “Jorge of Burgos wrote “The Name of the Rose”

# Formalizing the Puzzle in Sequent Calculus

- List of premises F = {p, not (r and q), q or (not p), not(not(r))}
- Simplifying, F = {p, not(r) or not(q), q or (not p), r}
- Question: Does F lead to a contradiction?
- $F \vdash$ false?

# Deriving the Contradiction

[p, not(r and q), q or (not(p)), r] ⊢ false

[p, q or (not(p)), r, not(r and q)] ⊢ (r and q)

q or (not(p)) ε F

[F | not(p)] ⊢ (r and q)

[F | q] ⊢ (r and q)

# Proof by Contradiction in Mathematics

- One of the most useful proof techniques
- Assume the *opposite* of what is to be proved
- Show that this assumption leads to a contradiction
- Therefore what is to be proved must be true

# Example: Euclid's Proof of the Infinitude of Primes

- Suppose there were only finitely many primes
- Then there must be a largest one, say $p_m$
- Now let $N = p_1 p_2 \dots p_m + 1$
- None of the primes $p_1 \dots p_m$ divides N
- Therefore, either N must be a prime itself or there must be a prime larger than $p_m$ dividing it
- Contradiction!

# Contradiction vs Paradox

- A contradiction is where a set of assumptions can be shown to be inconsistent, therefore at least one of the assumptions must be false
- In a paradox, no dubious assumptions are made, and yet a contradiction can be reached!

# Example: Epimenides' "Paradox"

- Epimenides' paradox: "All Cretans are liars"
- Epimenides is a Cretan. Is he a liar or is he telling the truth?
- If he is telling the truth, then since he is a Cretan, he must be lying. Contradiction!
- So Epimenides is a liar

# Example: Liar Paradox

- Eubulides' (Liar) paradox: "This sentence is false"
- Is the sentence false or true?
- If it is false, then "This sentence is false" is false, which means the sentence is true
- If it is true, then "This sentence is false" is true, which means it is false
- Paradox!

# Example: Russell's Paradox

- Gottlob Frege proposed formalization of mathematics in late 19th century, based on intuitive notion of a set
- Russell's paradox questions this intuitive notion
- Is the set of all sets which are not members of themselves a member of itself?
- Yes and no answers both lead to contradiction, just as with Liar Paradox

# Example: Berry's Paradox

- Consider "the smallest positive integer not definable in fewer than twelve words"
- I have just defined this integer in eleven words!

# Resolving Paradoxes

- Paradoxes show that “common-sense” ways of doing things can lead to logical difficulties
- Liar Paradox: A sentence should not be allowed to talk about its own truth value (self-reference)
- Russell’s Paradox: Care should be taken when defining sets of sets
- Berry’s Paradox: Not clear always when a phrase defines a number
- Paradoxes can be productive: Liar paradox is inspiration for Godel’s famous incompleteness theorem!

# Things to Practice

- Try some proofs with the full set of negation rules.