Probabilistic FSMs

In this lecture we look briefly at probabilistic finite state systems and hidden markov models.

Types of FSMs

- Deterministic: Single path through transition network determined by input symbols.
- Non-deterministic: Multiple paths possible with the same sequence of input symbols.
- Probablistic: Multiple paths possible but assume a random process choosing the paths.

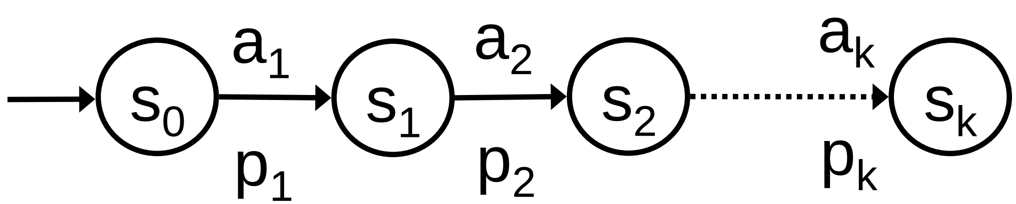
Formal Definition

Probabilistic FSM model, $M = (Q,\Sigma,s_0,F,\Delta)$

- Set of states, Q (one identified as initial)
- Set of input symbols, Σ (“input alphabet”)
- Initial state, $s_0 \in Q$
- Set of accepting states, $F \subseteq Q$
- Transition relation, Δ, that can generate the set of successor states given the current state and the set of transitions, T, each of the form (s, a, s′)
- A probability p(s, a, s′) associated with each transition, such that $\Sigma_{s'}$ p(s, a, s′) = 1 for each s that has transitions.

Probability of a Path

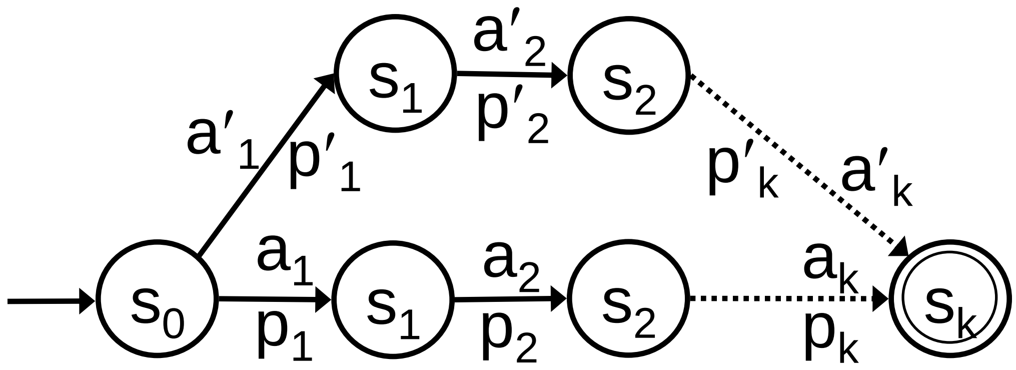
For a trace for string $a_1a_2\ldots a_k$ with probabilities on transitions $p_1p_2\ldots p_k$



$$p\_trace(a_1a_2\ldots a_k) = p_1 \times p_2 \times \ldots \times p_k$$

Take the product because the trace requires that the steps in the trace occur conjunctively

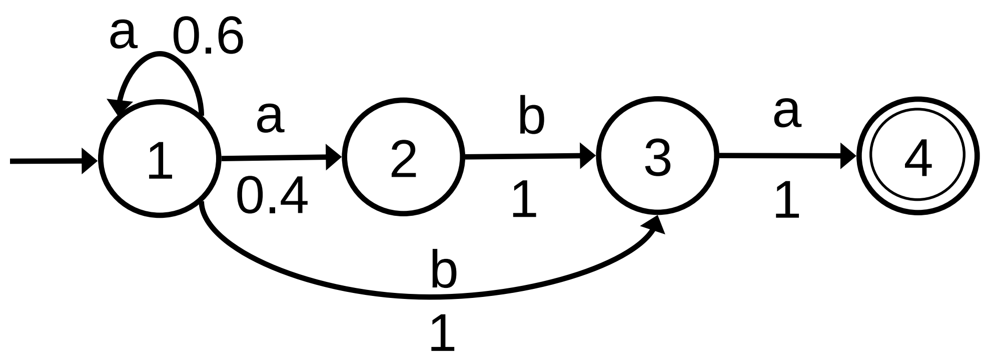
Probability of Acceptance



$$p\_accept(a_1 a_2 \ldots a_k) = \Sigma\, p\_trace(a_1 a_2 \ldots a_k)$$

Take the sum because acceptance is disjunctive across the traces

Example



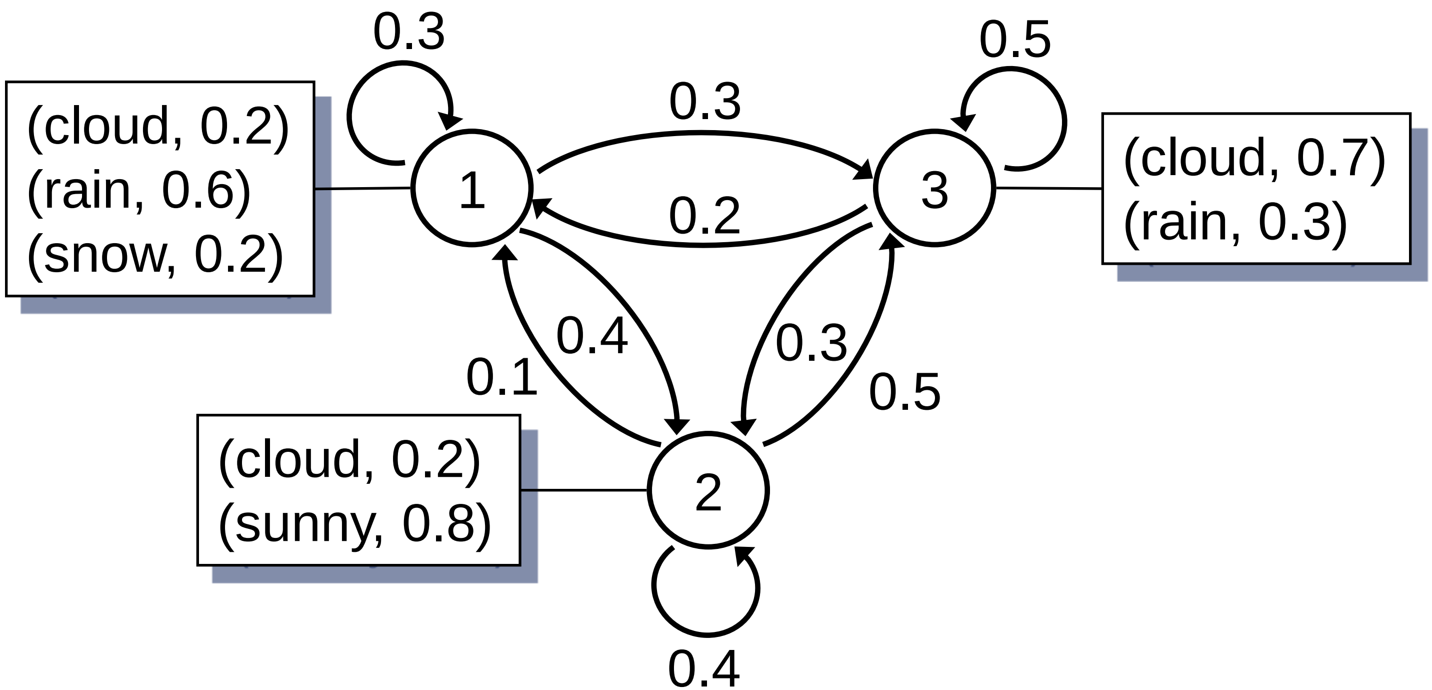
$$
\begin{aligned}
\text{p\_accept(aaba)} &= \Sigma\ \text{p\_trace(aaba)} \\
&= (0.6\times0.4\times1\times1) + (0.6\times0.6\times1\times1) \\
&= 0.24 + 0.36 \\
&= 0.6
\end{aligned}
$$

Hidden Markov Models

Adapted probabilistic transducer model, $M = (Q,\Delta)$

- Set of states, Q, each state of the form $(s,\{(o_1,p_1),\ldots(o_n,p_n)\})$ where o_i is an output observed at state s and p_i is the probability of observing that output when at that state.
- Transition relation, Δ, that can generate the set of successor states given the current state and the set of transitions, T, each of the form (s, s′). Note that transitions have no labels.
- A probability p(s, a, s′) associated with each transition, such that $\Sigma_{s'}$ p(s, s′) = 1 for each s that has transitions.

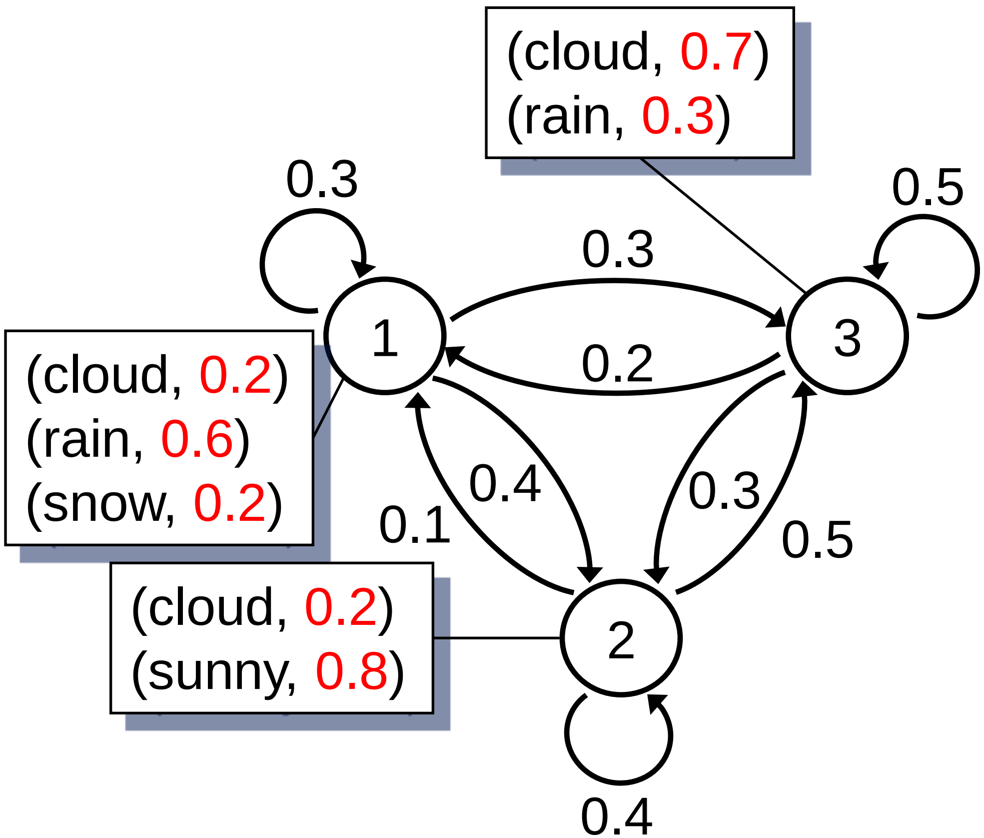
Example: Changes in Weather



Supposing that we start in state 1, what is the probability of observing the sequence: “cloud rain cloud sun” ?

Example:

Starting in state 1, what is the probability of observing the sequence: “cloud rain cloud sun” ?



Probabilities on traces	Product	Sum
$1 \xrightarrow{0.3} 1 \xrightarrow{0.3} 1 \xrightarrow{0.4} 2$; 0.2, 0.6, 0.2, 0.8	0.0007	0.0054
$1 \xrightarrow{0.3} 1 \xrightarrow{0.3} 3 \xrightarrow{0.3} 2$; 0.2, 0.6, 0.7, 0.8	0.0018	
$1 \xrightarrow{0.3} 1 \xrightarrow{0.4} 2 \xrightarrow{0.4} 2$; 0.2, 0.6, 0.2, 0.8	0.0009	
$1 \xrightarrow{0.3} 3 \xrightarrow{0.5} 3 \xrightarrow{0.3} 2$; 0.2, 0.3, 0.7, 0.8	0.0015	
$1 \xrightarrow{0.3} 3 \xrightarrow{0.3} 2 \xrightarrow{0.4} 2$; 0.2, 0.3, 0.2, 0.8	0.0003	
$1 \xrightarrow{0.3} 3 \xrightarrow{0.2} 1 \xrightarrow{0.4} 2$; 0.2, 0.3, 0.2, 0.8	0.0002	