Informatics 1A

Computation and Logic 9

DPLL (an idea)

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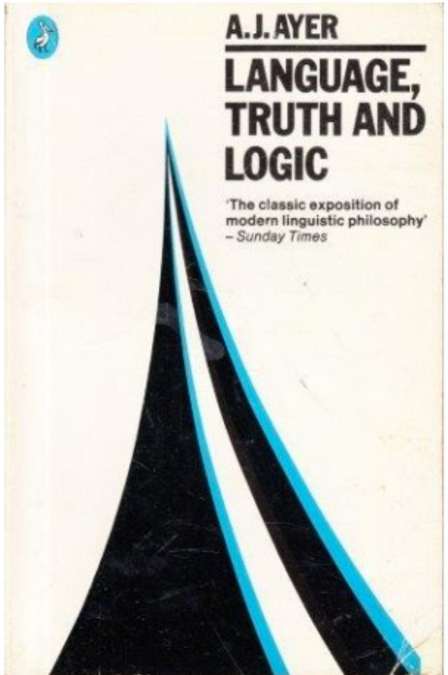
searching
for satisfaction



```haskell
-- Predicates: isGreen isBig isMortal isSocrates ...
Pred u :: u -> Bool

-- a universe of things
things :: [Thing]

-- a ⊨ b every a satisfies b
(|=) :: Pred Thing -> Pred Thing -> Bool
a |= b = and[ b x | x <- things, a x ]

-- logical operations on predicates
neg    :: Pred u -> Pred u
(&:&)  :: Pred u -> Pred u -> Pred u
(|:|)  :: Pred u -> Pred u -> Pred u
neg p   = (\x -> not(p x))
p &:& q = (\x -> p x && q x)
p |:| q = (\x -> p x || q x)
```

a $\qquad$ e $\qquad$ i $\qquad$ o

$$a \vDash b \qquad a \vDash \neg b \qquad a \nvDash \neg b \qquad a \nvDash b$$

If a, b are predicates in some universe, $a \vDash b$ iff *every a satisfies b;*
in this case we say the statement $a \vDash b$ is **valid**;
otherwise, the statement $a \vDash b$ is **invalid**, and the statement $a \nvDash b$ is valid.
We interpret $a \nvDash b$ as *some a is not b.*

$$\frac{m \vDash p \quad s \vDash m}{s \vDash p}\ \textit{barbara} \qquad \frac{p \vDash m \quad s \nvDash m}{s \nvDash p}\ \textit{baroco} \qquad \frac{m \nvDash p \quad m \vDash s}{s \nvDash p}\ \textit{bocardo}$$

$$\frac{m \vDash \neg p \quad s \vDash m}{s \vDash \neg p}\ \textit{celarent} \qquad \frac{p \vDash \neg m \quad s \nvDash \neg m}{s \nvDash p}\ \textit{festino} \qquad \frac{m \nvDash \neg p \quad m \vDash s}{s \nvDash \neg p}\ \textit{disamis}$$

$$\frac{p \vDash m \quad m \vDash \neg s}{s \vDash \neg p}\ \textit{calemes} \qquad \frac{p \vDash \neg m \quad m \nvDash \neg s}{s \nvDash p}\ \textit{fresison} \qquad \frac{p \nvDash \neg m \quad m \vDash s}{s \nvDash \neg p}\ \textit{dimatis}$$

$$\frac{p \vDash \neg m \quad s \vDash m}{s \vDash \neg p}\ \textit{cesare} \qquad \frac{m \vDash \neg p \quad s \nvDash \neg m}{s \nvDash p}\ \textit{ferio} \qquad \frac{m \vDash p \quad m \nvDash \neg s}{s \nvDash \neg p}\ \textit{datisi}$$

$$\frac{p \vDash m \quad s \vDash \neg m}{s \vDash \neg p}\ \textit{camestres} \qquad \frac{m \vDash \neg p \quad m \nvDash \neg s}{s \nvDash p}\ \textit{ferison} \qquad \frac{m \vDash p \quad s \nvDash \neg m}{s \nvDash \neg p}\ \textit{darii}$$

We extend the definition of $\models$ to allow a finite set of predicates on either side of the turnstile

$$\Gamma \models \Delta$$

. We define validity for these *sequents* in terms of the relation given earlier for individual predicates.

$$\Gamma \models \Delta \quad \texttt{iff} \quad \bigwedge \Gamma \models \bigvee \Delta$$

Here, $\bigwedge, \bigvee$ are the functions, `bigAnd` and `bigOr`, that give the conjunction and disjunction of a finite set of predicates. In Haskell,

```
bigAnd gamma = (\x -> and[ g x | g <- gamma ])
bigOr  delta = (\x ->  or[ d x | d <- delta ])
```

If `things` is a list of every thing in the universe, we can define

```
gamma |= delta = and[ or [ d x | d <- delta ]
                     | x <- things, and[ g x | g <- gamma ] ]
```

every thing that satisfies all predicates $g \in \Gamma$ satisfies some predicate $d \in \Delta$.

- a, b are predicates in some universe; Γ, Δ are finite sets of predicates,
- Γ, a refers to $\Gamma \cup \{a\}$; b, Δ refers to $\{b\} \cup \Delta$.
- Each of these rules is sound in both directions: all of the statements above the inference lines are valid iff all of the statements below the lines are valid.

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\frac{}{\Gamma, \bot \vDash \Delta}\ (\bot L) \qquad \frac{\Gamma \vDash \Delta}{\Gamma \vDash \bot, \Delta}\ (\bot R)$$

$$\frac{\Gamma \vDash \Delta}{\Gamma, \top \vDash \Delta}\ (\top L) \qquad \frac{}{\Gamma \vDash \top, \Delta}\ (\top R)$$

$$\frac{}{\Gamma, a \vDash \Delta, a}\ (I)$$

$$\frac{\Gamma, a, b \vDash \Delta}{\Gamma, a \wedge b \vDash \Delta}\ (\wedge L) \qquad \frac{\Gamma \vDash a, b, \Delta}{\Gamma \vDash a \vee b, \Delta}\ (\vee R)$$

$$\frac{\Gamma, a \vDash \Delta \quad \Gamma, b \vDash \Delta}{\Gamma, a \vee b \vDash \Delta}\ (\vee L) \qquad \frac{\Gamma \vDash a, \Delta \quad \Gamma \vDash b, \Delta}{\Gamma \vDash a \wedge b, \Delta}\ (\wedge R)$$

$$\frac{\Gamma \vDash a, \Delta}{\Gamma, \neg a \vDash \Delta}\ (\neg L) \qquad \frac{\Gamma, a \vDash \Delta}{\Gamma \vDash \neg a, \Delta}\ (\neg R)$$

$$\dfrac{\dfrac{}{\neg a, \neg c \vee b \vDash \neg a, c} \qquad \dfrac{\dfrac{\dfrac{a, b \vDash c}{b, \vDash \neg a, c}}{b, \neg c \vDash \neg a, c} \qquad \dfrac{a, b \vDash c}{b, b \vDash \neg a, c}}{b, \neg c \vee b \vDash \neg a, c}}{\dfrac{\dfrac{\neg a \vee b, \neg c \vee b \vDash \neg a, c}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}}{\dfrac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)), (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}}}$$

Our two inference trees tell two different stories …

$$\overline{p \vDash q,\, p}$$
$$\frac{p \vDash q,\, p}{\vDash \neg p,\, q,\, p}$$
$$\frac{\vDash \neg p,\, q,\, p}{\vDash \neg p \vee q,\, p} \qquad \frac{\overline{p \vDash p}}{\vDash \neg p,\, p}$$
$$\frac{\vDash \neg p \vee q,\, p \qquad \vDash \neg p,\, p}{\vDash (\neg p \vee q) \wedge \neg p,\, p}$$
$$\frac{\vDash (\neg p \vee q) \wedge \neg p,\, p}{\vDash ((\neg p \vee q) \wedge \neg p) \vee p}$$

$$\frac{a, b \vDash c}{b, \vDash \neg a,\, c}$$
$$\frac{b, \vDash \neg a,\, c}{b,\, \neg c \vDash \neg a,\, c} \qquad \frac{a, b \vDash c}{b,\, b \vDash \neg a,\, c}$$
$$\overline{\neg a,\, \neg c \vee b \vDash \neg a,\, c} \qquad \frac{b,\, \neg c \vDash \neg a,\, c \qquad b,\, b \vDash \neg a,\, c}{b,\, \neg c \vee b \vDash \neg a,\, c}$$
$$\frac{\neg a,\, \neg c \vee b \vDash \neg a,\, c \qquad b,\, \neg c \vee b \vDash \neg a,\, c}{\neg a \vee b,\, \neg c \vee b \vDash \neg a,\, c}$$
$$\frac{\neg a \vee b,\, \neg c \vee b \vDash \neg a,\, c}{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}$$
$$\frac{(\neg a \vee b) \wedge (\neg c \vee b) \vDash \neg a \vee c}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\, (\neg a \vee c)}$$
$$\frac{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)),\, (\neg a \vee c)}{\vDash \neg((\neg a \vee b) \wedge (\neg c \vee b)) \vee (\neg a \vee c)}$$

Every branch is terminated by an immediate rule.

The sequent we started from is valid in every universe!

Some branches lead to *leaves*, sequences with only atoms, in which no atom occurs on both sides of the turnstile.

Our starting sequent is valid in some universe U iff each of these leaves is valid.

It is easy to construct a counterexample to any one of these leaves.

Reduction using Gentzen Rules

show universal validity, or
provide counterexamples

compute L/R rules for other connectives

derive boolean equations

convert to CNF

Magic!

Boolean Algebra

$x \lor (y \lor z) = (x \lor y) \lor z$	$x \land (y \land z) = (x \land y) \land z$	associative
$x \lor (y \land z) = (x \lor y) \land (x \lor z)$	$x \land (y \lor z) = (x \land y) \lor (x \land z)$	distributive
$x \lor y = y \lor x$	$x \land y = y \land x$	commutative
$x \lor 0 = x$	$x \land 1 = x$	identity
$x \lor 1 = 1$	$x \land 0 = 0$	annihilation
$x \lor x = x$	$x \land x = x$	idempotent
$x \lor \neg x = 1$	$\neg x \land x = 0$	complements
$x \lor (x \land y) = x$	$x \land (x \lor y) = x$	absorbtion
$\neg(x \lor y) = \neg x \land \neg y$	$\neg(x \land y) = \neg x \lor \neg y$	de Morgan
$\neg\neg x = x$	$x \rightarrow y = \neg x \leftarrow \neg y$	

Reduction using Gentzen Rules

show universal validity, or
provide counterexample

compute L/R rules for other connectives

convert to CNF

derive Boolean equations

$$\frac{?}{\models a \wedge \neg a}$$

It is easy to find a counterexample

$$\frac{\models a \qquad \models \neg a}{\models a \wedge \neg a}$$

— but can we find an example?

Here we can easily see there is
no valuation
that makes both premises valid.

Other cases may not be so simple.

7		8				3		
			2		1			
5								
	4						2	6
3				8				
			1				9	
	9		6					4
				7		5		

a clause is a disjunction of literals
Or lits
a Form is a conjunction of clauses
And cs
a literal is N a or P a
where a is an atom

Does this sudoku problem have a solution?

Can we find a solution?

We will produce a CNF
sudoku = And rs
that *expresses the rules*
and a CNF
problem = And ps
that *represents the problem*

such that an example of
And (rs ++ ps)
is a *solution to the problem*

sudoku is a toy problem

7		8				3		
			2		1			
5								
	4						2	6
3				8				
			1				9	
	9		6					4
				7		5		

we will give an algorithm,
a version of DPLL (1962)

on modern hardware this
can solve sudoku problems
with 10 Ki clauses

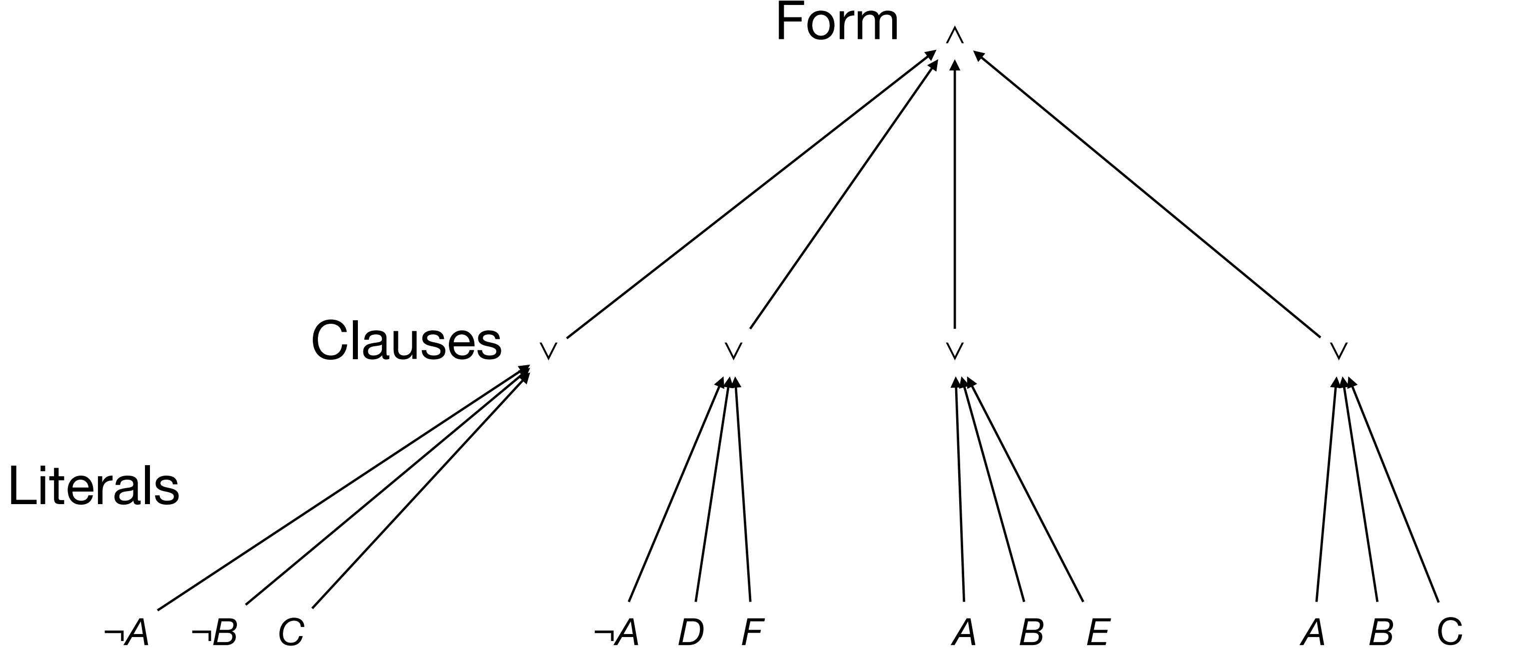
modern SAT solvers can
handle problems with
10 Mi clauses

the general problem is
Boolean satisfiability SAT

Is there a state that
satisfies a given CNF ?

practical applications include

verification of
hardware, software,
finite state machines,
communication protocols
…
AI planning
…
genomics
…



$$\models \neg A, \neg B, C \qquad \models \neg A, D, F \qquad \models A, B, E \qquad \models A, B, \neg C$$

$$\models \neg A, D, F \qquad \models A, B, E$$

$$\models \neg A, \neg B, C \qquad \ldots \models \ldots \qquad \ldots \models \ldots \qquad \ldots \models \ldots \qquad \models A, B, C$$

$$\ldots \models \ldots \qquad \ldots \models \ldots \qquad \ldots \models \ldots \qquad \ldots \models \ldots$$

$$\ldots \models \ldots \qquad \ldots \models \ldots$$

$$\Gamma \models \Delta$$

```haskell
data Literal a = P a | N a
newtype Clause a = Or [ Literal a ]
newtype Form a   = And[ Clause a ]

neg :: Literal a -> Literal a
neg (P a) = N a
neg (N a) = P a

data Atom = A|B|C|D|W|X|Y|Z deriving Eq

eg = And[ Or[N A, N C, P D], Or[P A, P C], Or[N D] ]
--          (¬A ∨ ¬C ∨ D)  ∧    (A ∨ C)    ∧    ¬D

type Val a = [ Literal a ]
```

Searching for a consistent set of literals, Γ

such that

$$\Gamma \models \neg A, \neg B, C \qquad \Gamma \models \neg A, D, F \qquad \Gamma \models A, B, E \qquad \Gamma \models A, B, \neg C$$

we say such a Γ is a **model** of the CNF

Divide and conquer

a problem shared is a problem (almost) solved

What if A is one of our literals?

$$\models \neg A, \neg B, C \quad \models \neg A, D, F \quad \models A, B, E \quad \models A, B, \neg C$$

$$\text{Searching for a consistent set of literals, } \Gamma$$

$$\textit{such that}$$

$$\Gamma \models \neg A, \neg B, C \qquad \Gamma \models \neg A, D, F \qquad \Gamma \models A, B, E \qquad \Gamma \models A, B, \neg C$$

Divide and conquer

a problem shared is a problem (almost) solved

What if A is one of our literals?

$$\frac{?}{A, \Gamma \models \neg A, \neg B, C} \quad \frac{?}{A, \Gamma \models \neg A, D, F} \quad \frac{?}{A, \Gamma \models A, B, E} \quad \frac{?}{A, \Gamma \models A, B, \neg C}$$

$$\text{Searching for a consistent set of literals, } \Gamma$$

$$\textit{such that}$$

$$\Gamma \models \neg A, \neg B, C \qquad \Gamma \models \neg A, D, F \qquad \Gamma \models A, B, E \qquad \Gamma \models A, B, \neg C$$

Divide and conquer

a problem shared is a problem (almost) solved

What if A is one of our literals?

$$\frac{A, \Gamma \models \neg B, C}{A, \Gamma \models \neg A, \neg B, C} \quad \frac{A, \Gamma \models D, F}{A, \Gamma \models \neg A, D, F} \quad \frac{}{A, \Gamma \models A, B, E} \quad \frac{}{A, \Gamma \models A, B, \neg C}$$

$$\text{Searching for a consistent set of literals, } \Gamma$$

$$\textit{such that}$$

$$\Gamma \models \neg A, \neg B, C \qquad \Gamma \models \neg A, D, F \qquad \Gamma \models A, B, E \qquad \Gamma \models A, B, \neg C$$

Divide and conquer

a problem shared is a problem (almost) solved

What if A is one of our literals?

$$\dfrac{\dfrac{\Gamma \models \neg B, C}{A, \Gamma \models \neg B, C}}{A, \Gamma \models \neg A, \neg B, C} \quad \dfrac{\dfrac{\Gamma \models D, F}{A, \Gamma \models D, F}}{A, \Gamma \models \neg A, D, F} \quad \dfrac{}{A, \Gamma \models A, B, E} \quad \dfrac{}{A, \Gamma \models A, B, \neg C}$$

Searching for a consistent set of literals, Γ

such that

$$\Gamma \models \neg A, \neg B, C \qquad \Gamma \models \neg A, D, F \qquad \Gamma \models A, B, E \qquad \Gamma \models A, B, \neg C$$

Divide and conquer

a problem shared is a problem (almost) solved

What if ¬A is one of our literals?

$$\frac{?}{A \models \neg A, \neg B, C} \quad \frac{?}{A \models \neg A, D, F} \quad \frac{?}{\neg A \models A, B, E} \quad \frac{?}{\neg A \models A, B, \neg C}$$

$$\text{Searching for a consistent set of literals, } \Gamma$$

$$\textit{such that}$$

$$\Gamma \vDash \neg A, \neg B, C \qquad \Gamma \vDash \neg A, D, F \qquad \Gamma \vDash A, B, E \qquad \Gamma \vDash A, B, \neg C$$

Divide and conquer

a problem shared is a problem (almost) solved

What if ¬A is one of our literals?

$$\frac{}{\neg A, \Gamma \vDash \neg A, \neg B, C} \qquad \frac{}{\neg A, \Gamma \vDash \neg A, D, F} \qquad \frac{\dfrac{\Gamma \vDash B, E}{\neg A, \Gamma \vDash B, E}}{\neg A, \Gamma \vDash A, B, E} \qquad \frac{\dfrac{\Gamma \vDash B, \neg C}{\neg A, \Gamma \vDash B, \neg C}}{\neg A, \Gamma \vDash A, B, \neg C}$$

$$\dfrac{\dfrac{\Gamma \models \neg B, C}{A, \Gamma \models \neg B, C}}{A, \Gamma \models \neg A, \neg B, C} \quad \text{if A} \quad \dfrac{\dfrac{\Gamma \models D, F}{A, \Gamma \models D, F}}{A, \Gamma \models \neg A, D, F} \quad \dfrac{}{A, \Gamma \models A, B, E} \quad \dfrac{}{A, \Gamma \models A, B, \neg C}$$

```
models [Clause Atom] -> [Val Atom]
models ([ Or[ N A, N B, P C ], Or[ N A, P D, P F ]
        , Or[ P A, P B, P E ], Or[ P A, P B, N C ] ] )
= [ P A : m | m <- models [ Or[ N B, P C ], Or[ P D, P F ] ]
  ++
  [ N A : m | m <- models [ Or[ P B, P E ], Or[ P B, N C ] ]
```

Tomorrow we will turn this idea into an algorithm

$$\dfrac{}{\neg A, \Gamma \models \neg A, \neg B, C} \quad \dfrac{}{\neg A, \Gamma \models \neg A, D, F} \quad \dfrac{\dfrac{\Gamma \models B, E}{\neg A, \Gamma \models B, E}}{\neg A, \Gamma \models A, B, E} \quad \text{if } \neg\text{A} \quad \dfrac{\dfrac{\Gamma \models B, \neg C}{\neg A, \Gamma \models B, \neg C}}{\neg A, \Gamma \models A, B, \neg C}$$