SAT and SMT algorithms

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Basic question

Given a propositional logic formula, is it satisfiable?
Standard to always put formulas into Conjunctive Normal Form or CNF.

- By introducing new variables this can be done with only constant-factor growth in formula size.

Terminology

- An atom p is a propositional symbol
- A literal l is an atom p or the negation of an atom $\neg p$.
- A clause C is a disjunction of literals $l_1 \vee \ldots \vee l_n$.
- A CNF formula F is a conjunction of clauses $C_1 \wedge \ldots \wedge C_m$

Abstract rules for DPLL

Core algorithms used in SAT and SMT solvers derived from DPLL algorithm (Davis,Putnam,Logemann,Loveland) from 1962.

Here present algorithms using abstract rule-based system due to Nieuwenhuis, Oliveras and Tinelli.

- General structure of algorithms easy to see
- Can work through simple examples on paper

General approach

- Try to incrementally build a satisfying truth assignment M for a CNF formula F
- Grow M by
  - guessing truth value of a literal not assigned in M
  - deducing truth value from current M and F.
- If reach a contradiction ($M \models \neg C$ for some $C \in F$), undo some assignments in M and try starting to grow M again in a different way.
- If all variables from M assigned and no contradiction, a satisfying assignment has been found for F
- If exhaust possibilities for M and no satisfying assignment is found, F is unsatisfiable

Assignments and States

States:

$$\textbf{fail} \quad \text{or} \quad M \parallel F$$

where

- M is sequence of literals and decision points $\bullet$ denoting a partial truth assignment
- F is a set of clauses denoting a CNF formula

First literal after each $\bullet$ is called a decision literal

Decision points start suffixes of M that might be discarded when choosing new search direction

Def: If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is decision level i of M
- $M^{[i]} = M_0 \bullet \cdots \bullet M_i$

Initial and final states

Initial state

- $() \parallel F_0$

Expected final states

- **fail** if F_0 is unsatisfiable
- $M \parallel G$ otherwise, where
  - G is equivalent to F_0
  - M satisfies G

Classic DPLL rules

Decide

$$M \parallel F \Longrightarrow M \bullet l \parallel F \quad \textbf{if} \begin{cases} l \text{ or } \neg l \text{ in clause of } F, \\ l \text{ is undefined in } M \end{cases}$$

UnitPropagate

$$M \parallel F, C \vee l \Longrightarrow M\, l \parallel F, C \vee l \quad \textbf{if} \begin{cases} M \models \neg C, \\ l \text{ is undefined in } M \end{cases}$$

Fail

$$M \parallel F, C \Longrightarrow \textbf{fail} \quad \textbf{if} \begin{cases} M \models \neg C, \\ \bullet \notin M \end{cases}$$

Backtrack

$$M \bullet l\; N \parallel F, C \Longrightarrow M\, \neg l \parallel F, C \quad \textbf{if} \begin{cases} M \bullet l\; N \models \neg C \\ \bullet \notin N \end{cases}$$

Strategies for applying rules

- Are many heuristics for choosing literal l in Decide rule.
  - MOMS: choose literal with the Maximum number of Occurrences in Minimum Size clauses.
  - VSIDS: choose literal that has most frequently been involved in recent conflict clauses.

- UnitPropagate applied with higher priority than Decide since it does not introduce branching in search
  - Typically many UnitPropagate applications for each Decide
  - BCP (Boolean Constraint Propagation): repeated application of UnitPropagate

Strategies for applying rules (cont)

- After each Decide or UnitPropagate should check for a conflicting clause, a clause C for which

$$M \models \neg C \quad .$$

If there is a conflicting clause, Backtrack or Fail are applied immediately to avoid pointless search.

Example execution

	C_1		C_2		C_3		C_4			
M	$\bar{x}_1 \vee$	x_2	$\bar{x}_3 \vee$	x_4	$\bar{x}_5 \vee$	$\bar{x}_6$	$x_6 \vee$	$\bar{x}_5 \vee$	$\bar{x}_2$	Rule
$()$	u	u	u	u	u	u	u	u	u	
$\bullet x_1$	0	u	u	u	u	u	u	u	u	Decide x_1
$\bullet x_1 x_2$	0	1	u	u	u	u	u	u	0	UnitProp C_1
$\bullet x_1 x_2 \bullet x_3$	0	1	0	u	u	u	u	u	0	Decide x_3
$\bullet x_1 x_2 \bullet x_3 x_4$	0	1	0	1	u	u	u	u	0	UnitProp C_2
$\bullet x_1 x_2 \bullet x_3 x_4 \bullet x_5$	0	1	0	1	0	u	u	0	0	Decide x_5
$\bullet x_1 x_2 \bullet x_3 x_4 \bullet x_5 \bar{x}_6$	0	1	0	1	0	1	0	0	0	UnitProp C_3
$\bullet x_1 x_2 \bullet x_3 x_4 \bar{x}_5$	0	1	0	1	1	u	u	1	0	Backtrack C_4
$\bullet x_1 x_2 \bullet x_3 x_4 \bar{x}_5 \bar{x}_6$	0	1	0	1	1	1	0	1	0	Decide $\bar{x}_6$

- Last state here is final – no further rules apply
- Derivation shows that $C_1 \wedge C_2 \wedge C_3 \wedge C_4$ is satisfiable
- Final M is a satisfying assignment

Implication graphs

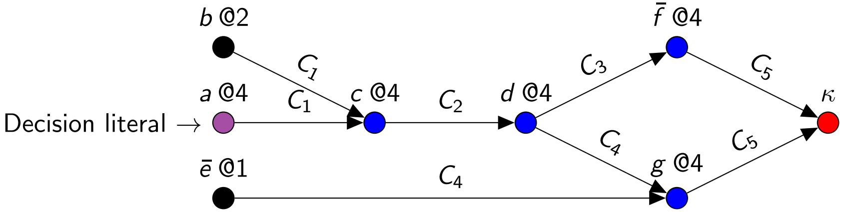
An implication graph describes the dependencies between literals in an assignment

- 1 node per assigned literal
  - Node label $l\,@i$ indicates literal l is assigned true at decision level i.
- Roots of graph (nodes without in-edges) are literals in M_0 and decision literals
- Edges $l_1 \rightarrow l, \cdots, l_n \rightarrow l$ added if unit propagation with clause $\neg l_1 \vee \cdots \vee \neg l_n \vee l$ sets literal l
  - Each edge labelled with clause
- When current assignment is conflicting with conflicting clause $\neg l_1 \vee \cdots \vee \neg l_n$, then conflict node κ and edges $l_1 \rightarrow l, \cdots, l_n \rightarrow l$ are added
  - Each edge labelled with conflicting clause

Partial Implication graph example

Only shows current decision-level nodes and immediately-preceding nodes.

$$C_1 = \bar{a} \vee \bar{b} \vee c \quad C_2 = \bar{c} \vee d \quad C_3 = \bar{d} \vee \bar{f}$$
$$C_4 = \bar{d} \vee e \vee g \quad C_5 = f \vee \bar{g}$$



Backjump clause inference

The implication graph enables inference of new clauses entailed by the current formula F and made false by the current assignment.

- Consider any cut of an implication graph with
  - On right: conflicting node κ
  - On left: decision literal for current level and all literals at lower levels
- If literals on immediate left of cut are $l_1, \ldots, l_n$, then can infer the new clause

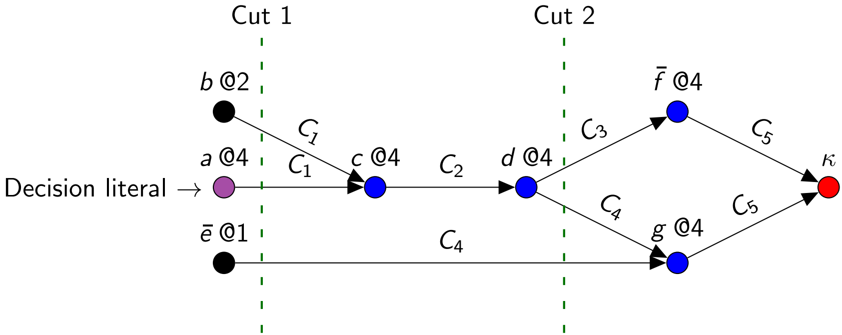
  $$(l_1 \wedge \cdots \wedge l_n) \Rightarrow \textbf{false}$$

  or equivalently

  $$\neg l_1 \vee \cdots \vee \neg l_n$$

Clause inference example

$$C_1 = \bar{a} \vee \bar{b} \vee c \quad C_2 = \bar{c} \vee d \quad C_3 = \bar{d} \vee \bar{f}$$
$$C_4 = \bar{d} \vee e \vee g \quad C_5 = f \vee \bar{g}$$



Backjump clause: $\bar{b} \vee \bar{a} \vee e$ $\quad$ $\bar{d} \vee e$

Backjumping

If

- current assignment has form $M \bullet l\ N$, and
- the inferred clause has form $C' \vee l'$ where l' is the only literal at the current decision level, and
- all literals of C' are assigned in M,

then it is legitimate to

- backjump, set the assignment to M, and
- noting that $C' \vee l'$ has exactly one literal unassigned in M, to apply unit propagation to extend the assignment to $M\ l'$.

Such a clause $C' \vee l'$ is called a backjump clause

A backjump clause can always be formed using the decision literal from the current level

Smaller backjump clauses can sometimes be discovered that exploit unique implication points (UIPs), literals on every path from the current decision literal to the conflict node κ.

Backjump rule

Replaces and generalises Backtrack rule in modern DPLL implementations

Backjump

$$M \bullet l\ N \ \| \ F, C \Longrightarrow M\ l' \ \| \ F, C \ \ \textbf{if} \begin{cases} M \bullet l\ N \models \neg C, \text{ and there is some clause } C' \vee l' \text{ such that:} \\ -\ F, C \models C' \vee l', \\ -\ M \models \neg C', \\ -\ l' \text{ is undefined in } M, \text{ and} \\ -\ l' \text{ or } \neg l' \text{ occurs in } F \text{ or in } M \bullet l\ N \end{cases}$$

- C is the conflicting clause
- $C' \vee l'$ is the backjump clause

Learning

Learn

$$M \parallel F \Longrightarrow M \parallel F, C \;\; \textbf{if} \begin{cases} \text{each atom of } C \text{ occurs in} \\ F \text{ or in } M, \\ F \models C \end{cases}$$

- Common C are backjump clauses from the Backjump rule.
- Learned clauses record information about parts of search space to be avoided in future search
- CDCL (Conflict Driven Clause Learning) = Backjump + Learn

Forgetting

Forget

$$M \parallel F, C \Longrightarrow M \parallel F \ \ \textbf{if} \ F \models C$$

- Applied to C considered less important.
- Essential for controlling growth of required storage.
- Performance can degrade as F grows, so shrinking F can improve performance.

Restarting

Restart

$$M \parallel F \Longrightarrow () \parallel F$$

- Only used if F grown using learning.
- Additional knowledge causes Decide heuristics to work differently and often explore search space in more compact way.
- To preserve completeness, applied repeatedly with increasing periodicity.

Why is DPLL correct? 1

Lemma (1 - nature of reachable states)

Assume $() \parallel F \Longrightarrow^* M \parallel F'$. *then*

1. F *and* F' *are equivalent*
2. *If* M *is of the form* $M_0 \bullet l_1 M_1 \cdots \bullet l_n M_n$ *where all* M_i *are* $\bullet$ *free, then* $F, l_1, \ldots l_i \models M_i$ *for all* i *in* $0 \ldots n$.

Lemma (2 - nature of final states)

If $() \parallel F \Longrightarrow^* S$ *and* S *is final (no further transitions possible), then either*

1. $S =$ **fail**, *or*
2. $S = M \parallel F'$ *where* $M \models F$

Why is DPLL correct? 2

Lemma (3 - transition sequences never go on for ever)

Every derivation $() \parallel F \Longrightarrow S_1 \Longrightarrow S_2 \Longrightarrow \cdots$ *is finite*

Proof.

Given M of form $M_0 \bullet M_1 \cdots \bullet M_n$ where all M_i are $\bullet$ free, define the *rank of* M, $\rho(M)$ as $\langle r_0, r_1, \ldots, r_n \rangle$ where $r_i = |M_i|$. Every derivation must be finite as each basic DPLL rule strictly increases the rank in a lexicographic order and the image of ρ is finite. □

Why is DPLL correct? 3

Theorem (1 - termination in fail state)

If $() \parallel F \Longrightarrow^* S$ *and* S *is final, then*

1. *if* S *is* **fail**, *then* F *is unsatisfiable*
2. *if* F *is unsatisfiable then* S *is* **fail**

Why is DPLL correct? 4

Proof.

1. We have $() \parallel F \Longrightarrow^* M \parallel F' \Longrightarrow$ **fail**.

   By Fail rule definition, there is a $C \in F'$ s.t. $M \models \neg C$.

   Since M is $\bullet$ free, we have by Lemma 1(2) that $F \models M$, and therefore $F \models \neg C$.

   However, $F' \models C$ and by Lemma 1(1) $F \models C$.

   Hence, F must be unsatisfiable.

2. By Lemma 2.

$\square$

Abstract DPLL modulo theories

Start just with one theory T. E.g.

- Equality with uninterpreted functions
- Linear arithmetic over Z or R.

Propositional atoms now both

- Propositional symbols
- Atomic relations over T involving individual expressions.
  E.g. $f(g(a)) = b$ or $3a + 5b \leq 7$.

Previous rules (e.g. Decide, UnitPropagate) and $\models$ (propositional entailment) treat syntactically distinct atoms as distinct

New rules involve $\models_T$ (entailment in theory T)

Theory learning

T-Learn

$$M \parallel F \Longrightarrow M \parallel F, C \quad \textbf{if} \begin{cases} \text{each atom of } C \text{ occurs in} \\ F \text{ or in } M, \\ F \models_T C \end{cases}$$

- One use is for catching when M is inconsistent from T point of view.
  - Say $\{l_1, \ldots, l_n\} \subseteq M$ such that $F \models_T l_1 \wedge \cdots \wedge l_n \Rightarrow$ **false**
  - Then add $C = \neg l_1 \vee \cdots \vee \neg l_n$
  - As C is conflicting, the Backjump or Fail rule is enabled
  - Theory solvers can identify unsat cores, small subsets of literals sufficient for creating a conflicting clause
- Frequency of checks $F \models_T C$ needs careful regulation, as cost might be far higher than basic DPLL steps.
- Given size of F often just check $\models_T C$. In this case C is called a theory lemma.

Theory propagation

Guiding growth of M rather than just detecting when it is T-inconsistent.

TheoryPropagate

$$M \;\|\; F \Longrightarrow M\, l \;\|\; F \quad \textbf{if} \begin{cases} M \models_T l, \\ l \text{ or } \neg l \text{ occurs in } F \\ l \text{ is undefined in } M \end{cases}$$

- If applied well, can dramatically increase performance
- Worth applying exhaustively in some cases before resorting to Decide

Integration of SAT and theory solvers

Use of T-Learn and TheoryPropagate rules requires close integration of SAT and theory solvers

- SAT solvers need modification to be able to call out to theory solvers
- Useful to have theory solvers incremental, able to be rerun efficiently when input is some small increment on previous input
  - Also need ability to efficiently retract blocks of input to cope with backjumping

Handling multiple theories

Consider formula F mixing theories of linear real arithmetic and uninterpreted functions:

$$\begin{gathered} f(x_1,0) \geq x_3 \ \wedge \ f(x_2,0) \leq x_3 \ \wedge \\ x_1 \geq x_2 \ \wedge \ x_2 \geq x_2 \ \wedge \\ x_3 - f(x_1,0) \geq 1 \end{gathered}$$

The popular Nelson-Oppen combination procedure involves first purifying, adding additional variables and creating an equisatisfiable formula with each atom over just one of the theories.

Formula F above is equisatisfiable with $F_1 \wedge F_2$, where

$$\begin{aligned} F_1 &= a_1 \geq x_3 \ \wedge \ a_2 \leq x_3 \ \wedge \ x_1 \geq x_2 \ \wedge \ x_2 \geq x_1 \ \wedge \\ &\quad\ x_3 - a_1 \geq 1 \ \wedge \ a_0 = 0 \\ F_2 &= a_1 = f(x_1, a_0) \ \wedge \ a_2 = f(x_2, a_0) \end{aligned}$$

F_1 just involves linear real arithmetic and F_2 just involves an uninterpreted function

Nelson-Oppen example

Separate theory solvers can work on F_1 and F_2, exchanging equalities

i	1	2
	R arith	EUF
Original F_i	$a_1 \geq x_3$	$a_1 = f(x_1, a_0)$
	$a_2 \leq x_3$	$a_2 = f(x_2, a_0)$
	$x_1 \geq x_2$	
	$x_2 \geq x_1$	
	$x_3 - a_1 \geq 1$	
	$a_0 = 0$	
Deduced atoms	$x_1 = x_2 (*)$	$x_1 = x_2$
	$a_1 = a_2$	$a_1 = a_2 (*)$
	$a_1 = x_3 (*)$	
	false $(*)$	

The $(*)$ marks indicate when inference is in the respective theory

Nelson-Oppen

The basic Nelson-Oppen procedure relies on combined theories being convex.

- Linear real arithmetic and EUF (Equality and Uninterpreted Functions) are convex.
- Linear integer arithmetic and bit-vector theories are not.

Extensions of Nelson-Oppen can handle a number of non-convex theories.

In general, a combination of decidable theories might be undecidable

Further reading

1. *A SAT Solver Primer*. David Mitchell. EATCS Bulletin (The Logic in Computer Science Column), Volume 85, February 2005.

2. *Efficient Conflict Driven Learning in a Boolean Satisfiability Solver*. L. Zhang, C. F. Madigan, M. H. Moskewicz and S. Malik. ICCAD 01:

3. *Solving SAT and SAT Modulo Theories: From an Abstract DavisPutnamLogemannLoveland Procedure to DPLL(T)* Robert Neiuwenhuis, Albert Oliveras, Cesare Tinelli. Journal of the ACM. 53(6):937-977, 2006

4. Slides and videos from the 2012 SAT/SMT Summer School `https://es-static.fbk.eu/events/satsmtschool12/`

These slides draw mainly on 3 and part of 2. Tinelli's presentation in 4 also expands on the Abstract DPLL approach to SAT and SMT.