FNLP Lecture 9: Algorithms for HMMs

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Based on slides by Henry Thompson and Sharon Goldwater

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Recap: HMM

- Elements of HMM:
  - Set of states (tags)
  - Output alphabet (word types)
  - Start state (beginning of sentence)
  - State transition probabilities
  - Output probabilities from each state

More general notation

- Previous lecture:
  - Sequence of tags $T = t_1 \ldots t_n$
  - Sequence of words $S = w_1 \ldots w_n$
- This lecture:
  - Sequence of states $Q = q_1 \ldots q_T$
  - Sequence of outputs $O = o_1 \ldots o_T$
  - So t is now a time step, not a tag! And T is the sequence length.

Recap: HMM

- Given a sentence $O = o_1 \ldots o_T$ with tags $Q = q_1 \ldots q_T$, compute $P(O,Q)$ as:

$$P(O,Q) = \prod_{t=1}^{T} P(o_t \mid q_t) P(q_t \mid q_{t-1})$$

- But we want to find $\operatorname{argmax}_Q P(Q \mid O)$ without enumerating all possible Q
  – Use Viterbi algorithm to store partial computations.

Today's lecture

- What algorithms can we use to
  - Efficiently compute the most probable tag sequence for a given word sequence?
  - Efficiently compute the likelihood for an HMM (probability it outputs a given sequence *s*)?
  - Learn the parameters of an HMM given unlabelled training data?
- What are the properties of these algorithms (complexity, convergence, etc)?

Tagging example

Words:

Possible tags:
(ordered by frequency for each word)

<s>	one	dog	bit	</s>
<s>	CD	NN	NN	</s>
	NN	VB	VBD	
	PRP			

Tagging example

Words:	<s>	one	dog	bit	</s>
Possible tags: (ordered by frequency for each word)	<s>	CD	NN	NN	</s>
		NN	VB	VBD	
		PRP			

- Choosing the best tag for each word independently gives the wrong answer (<s> CD NN NN </s>).
- P(VBD|bit) < P(NN|bit), but may yield a better *sequence* (<s> CD NN VB </s>)
  - because P(VBD|NN) and P(</s>|VBD) are high.

Viterbi: intuition

Words:

Possible tags: (ordered by frequency for each word)

<s>	one	dog	bit	</s>
<s>	CD	NN	NN	</s>
	NN	VB	VBD	
	PRP			

- Suppose we have already computed
  a) The best tag sequence for <s> … bit that ends in NN.
  b) The best tag sequence for <s> … bit that ends in VBD.
- Then, the best full sequence would be either
  – sequence (a) extended to include </s>, or
  – sequence (b) extended to include </s>.

Viterbi: intuition

Words:	<s>	one	dog	bit	</s>
Possible tags: (ordered by frequency for each word)	<s>	CD	NN	NN	</s>
		NN	VB	VBD	
		PRP			

- But similarly, to get
  a) The best tag sequence for <s> … bit that ends in NN.
- We could extend one of:
  – The best tag sequence for <s> … dog that ends in NN.
  – The best tag sequence for <s> … dog that ends in VB.
- And so on…

Viterbi: high-level picture

- Intuition: the best path of length t ending in state Q must include the best path of length t-1 to the previous state (call it P). (t now a *time*, not a *tag*):

$$\text{<s>/<s>} \quad \ldots \quad o_{t-1}/P \quad o_t/Q$$

- Because otherwise there must be a better path to P that we should have used, thereby getting a better path to Q
  - Remember the Markov assumptions
    - are *independent* of everything from <s> to Q

Viterbi: high-level picture

- Intuition: the best path of length t ending in state q must include the best path of length t-1 to the previous state. (t now a *time step*, not a *tag*). So,
  - Find the best path of length t-1 to each state.
  - Consider extending each of those by 1 step, to state q.
  - Take the best of those options as the best path to state q.

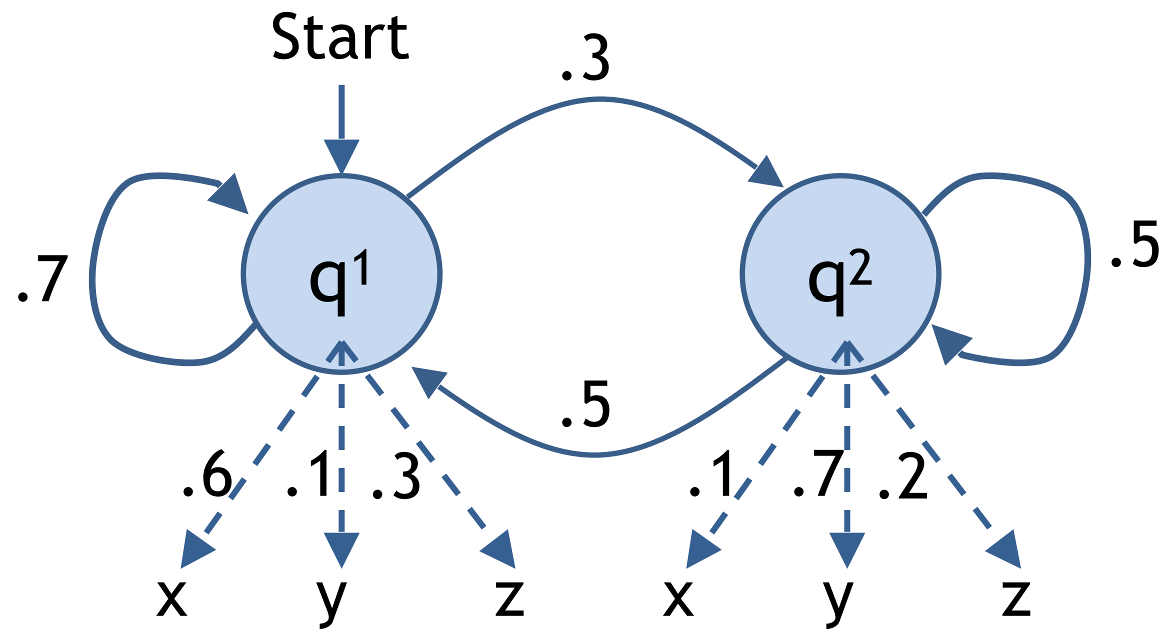
Notation

- Sequence of observations over time $o_1, o_2, \ldots, o_T$
  - here, words in sentence
- Vocabulary size V of possible observations
- Set of possible states $q^1, q^2, \ldots, q^N$ (see note next slide)
  - here, tags
- A, an $N \times N$ matrix of transition probabilities
  - a_{ij}: the prob of transitioning from state q^i to q^j. (JM3 Fig 8.7)
- B, an $N \times V$ matrix of output probabilities
  - $b_i(o_t)$: the prob of emitting o_t from state q^i. (JM3 Fig 8.8)

Note on notation

- J&M use $q_1, q_2, \ldots, q_N$ for set of states, but *also* use $q_1, q_2, \ldots, q_T$ for state sequence over time.
  - So, just seeing q_1 is ambiguous (though usually disambiguated from context).
  - I'll instead use q^i for state names, and q_t for state at time t.
  - So we could have $q_t = q^i$, meaning: the state we're in at time t is q^i.

HMM example w/ new notation



- States $\{q^1, q^2\}$ (or $\{<s>, q^1, q^2\}$)
- Output alphabet $\{x, y, z\}$

Adapted from Manning & Schuetze, Fig 9.2

Transition and Output Probabilities

- Transition matrix A:

  $a_{ij} = P(q^j \mid q^i)$

	q^1	q^2
<s>	1	0
q^1	.7	.3
q^2	.5	.5

- Output matrix B:

  $b_i(o) = P(o \mid q^i)$

  for output o

	x	y	z
q^1	.6	.1	.3
q^2	.1	.7	.2

Joint probability of (states, outputs)

- Let $\lambda = (A, B)$ be the parameters of our HMM.
- Using our new notation, given state sequence $Q = (q_1 \ldots q_T)$ and output sequence $O = (o_1 \ldots o_T)$, we have:

$$P(O, Q \mid \lambda) = \prod_{t=1}^{T} P(o_t \mid q_t) P(q_t \mid q_{t-1})$$

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- Or:

$$P(O, Q \mid \lambda) = \prod_{t=1}^{T} b_{q_t}(o_t)\, a_{q_{t-1} q_t}$$

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- Or:

$$P(O, Q \mid \lambda) = \prod_{t=1}^{T} b_{q_t}(o_t)\, a_{q_{t-1} q_t}$$

- Example:

$$P\Big(O = (y, z), Q = (q^1, q^1) \Big| \lambda\Big) \quad \begin{aligned} &= b1(y) \bullet b1(z) \bullet a_{<s>,1} \bullet a_{11} \\ &= (.1)(.3)(1)(.7) \end{aligned}$$

Viterbi: high-level picture

- Want to find $\text{argmax}_Q P(Q \mid O)$
- Intuition: the best path of length t ending in state q must include the best path of length $t\text{-}1$ to the previous state. So,
  - Find the best path of length $t\text{-}1$ to each state.
  - Consider extending each of those by 1 step, to state q.
  - Take the best of those options as the best path to state q.

Viterbi algorithm

- Use a **chart** to store partial results as we go
  - $N x T$ table, where $v(j,t)$ is the probability* of the best state sequence for $o_1 \ldots o_t$ that ends in state j.

*Specifically, $v(j,t)$ stores the max of the joint probability $P(o_1 \ldots o_t, q_1 \ldots q_{t-1}, q_t=j|\lambda)$

Viterbi algorithm

- Use a **chart** to store partial results as we go
  - NxT table, where $v(j,t)$ is the probability* of the best state sequence for $o_1 \ldots o_t$ that ends in state j.

- Fill in columns from left to right, with

$$v(j,t) = \max_{i=1}^{N} v(i,t-1) \bullet a_{ij} \bullet b_j(o_t)$$

*Specifically, v(j,t) stores the max of the joint probability $P(o_1 \ldots o_t, q_1 \ldots q_{t-1}, q_t = j | \lambda)$

Viterbi algorithm

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- Fill in columns from left to right, with

$$v(j,t) = \max_{i=1}^{N} v(i,t-1) \bullet a_{ij} \bullet b_j(o_t)$$

- Store a **backtrace** to show, for each cell, which state at t-1 we came from.

*Specifically, $v(j,t)$ stores the max of the joint probability $P(o_1 \ldots o_t, q_1 \ldots q_{t-1}, q_t = j | \lambda)$

Example

- Suppose O=xzy. Our initially empty table:

	o_1=x	o_2=z	o_3=y
q^1			
q^2			

Filling the first column

	o_1=x	o_2=z	o_3=y
q^1	.6		
q^2	0		

$$v(1,1) = a_{<s>1} \bullet b1\left(x\right) = (1)(.6)$$

$$v(2,1) = a_{<s>2} \bullet b2\left(x\right) = (0)(.1)$$

Starting the second column

	o_1=x	o_2=z	o_3=y
q^1	.6		
q^2	0		

$$v(1,2) = \max_{i=1}^{N} v(i,1) \bullet a_{i1} \bullet b1(z)$$

$$= \max \begin{cases} v(1,1) \bullet a_{11} \bullet b_1(z) = (.6)(.7)(.3) \\ v(2,1) \bullet a_{21} \bullet b_1(z) = (0)(.5)(.3) \end{cases}$$

Starting the second column

	o_1=x	o_2=z	o_3=y
q^1	.6	.126	
q^2	0		

$$v(1,2) = \max_{i=1}^{N} v(i,1) \bullet a_{i1} \bullet b1(z)$$

$$= \max \begin{cases} v(1,1) \bullet a_{11} \bullet b_1(z) = (.6)(.7)(.3) \\ v(2,1) \bullet a_{21} \bullet b_1(z) = (0)(.5)(.3) \end{cases}$$

Finishing the second column

	$o_1=x$	$o_2=z$	$o_3=y$
q^1	.6	.126	
q^2	0		

$$v(2,2) = \max_{i=1}^{N} v(i,1) \bullet a_{i2} \bullet b2(z)$$

$$= \max \begin{cases} v(1,1) \bullet a_{12} \bullet b_2(z) = (.6)(.3)(.2) \\ v(2,1) \bullet a_{22} \bullet b_2(z) = (0)(.5)(.2) \end{cases}$$

Finishing the second column

	o_1=x	o_2=z	o_3=y
q^1	.6	.126	
q^2	0	.036	

$$v(2,2) = \max_{i=1}^{N} v(i,1) \bullet a_{i2} \bullet b2(z)$$

$$= \max \begin{cases} v(1,1) \bullet a_{12} \bullet b_2(z) = (.6)(.3)(.2) \\ v(2,1) \bullet a_{22} \bullet b_2(z) = (0)(.5)(.2) \end{cases}$$

Third column

	$o_1=x$	$o_2=z$	$o_3=y$
q^1	.6	.126	.00882
q^2	0	.036	.02646

- Exercise: make sure you get the same results!

Best Path

	o_1=x	o_2=z	o_3=y
q^1	.6	.126	.00882
q^2	0	.036	.02646

- Choose best final state: $\max_{i=1}^{N} v(i,T)$
- Follow backtraces to find best full sequence: $q^1q^1q^2$

HMMs: what else?

- As with probabilities in N-gram models and classification, chart probabilities get really tiny really fast, risking underflow
  - So, we use **costs** (negative log probabilities) instead
  - Take minimum over sum of costs, instead of maximum over product of probabilities.
- Using Viterbi, we can find the best tags for a sentence (**decoding**), and get .
- We might also want to
  - Compute the **likelihood** , i.e., the probability of a sentence regardless of tags (a language model!)
  - **learn** the best set of parameters $\lambda = (A, B)$ given only an *unannotated* corpus of sentences.

Computing the likelihood

- From probability theory, we know that

$$P(O \mid \lambda) = \sum_{Q} P(O, Q \mid \lambda)$$

- There are an exponential number of Qs.
- Again, by computing and storing partial results, we can solve efficiently.
- (Next slides show the algorithm but I'll likely skip them)

Forward algorithm

- Use a table with cells $\alpha(j,t)$: the probability of being in state after seeing $o_1 \ldots o_t$ (**forward probability**).

$$\alpha(j,t) = P(o1, o2, \ldots ot, qt = j \mid \lambda)$$

- Fill in columns from left to right, with

$$\alpha(j,t) = \sum_{i=1}^{N} \alpha(i,t-1) \bullet a_{ij} \bullet b_j(o_t)$$

  - Same as Viterbi, but sum instead of max (and no backtrace).

Note: because there's a sum, we can't use the trick that replaces probabilitiess with costs. For implementation info, see http://digital.cs.usu.edu/~cyan/CS7960/hmm-tutorial.pdf and http://stackoverflow.com/questions/13391625/underflow-in-forward-algorithm-for-hmms

Example

- Suppose O=xzy. Our initially empty table:

	o_1=x	o_2=z	o_3=y
q^1			
q^2			

Filling the first column

	o_1=x	o_2=z	o_3=y
q^1	.6		
q^2	0		

$$\alpha(1,1) = a_{<s>1} \bullet b1(x) = (1)(.6)$$

$$\alpha(2,1) = a_{<s>2} \bullet b2(x) = (0)(.1)$$

Starting the second column

	o_1=x	o_2=z	o_3=y
q^1	.6	.126	
q^2	0		

$$\alpha(1,2) = \sum_{i=1}^{N} \alpha(i,1) \bullet a_{i1} \bullet b1(z)$$

$$= \alpha(1,1) \bullet a_{11} \bullet b_1(z) + \alpha(2,1) \bullet a_{21} \bullet b1(z)$$

$$= \left(.6\right)(.7)(.3) + (0)\left(.5\right)(.3)$$

$$= .126$$

Finishing the second column

	o_1=x	o_2=z	o_3=y
q^1	.6	.126	
q^2	0	.036	

$$\alpha(2,2) = \sum_{i=1}^{N} \alpha(i,1) \bullet a_{i2} \bullet b2(z)$$

$$= \alpha(1,1) \bullet a_{12} \bullet b_2(z) + \alpha(2,1) \bullet a_{22} \bullet b2(z)$$

$$= \left(.6\right)(.3)(.2) + (0)\left(.5\right)(.2)$$

$$= .036$$

Third column and finish

	o_1=x	o_2=z	o_3=y
q^1	.6	.126	.01062
q^2	0	.036	.03906

- Add up all probabilities in last column to get the probability of the entire sequence:

$$P(O \mid \lambda) = \sum_{i=1}^{N} \alpha(i, T)$$

Learning

- Given *only* the output sequence, learn the best set of parameters $\lambda = (A, B)$.
- Assume 'best' = maximum-likelihood.
- Other definitions are possible, won't discuss here.

Unsupervised learning

- Training an HMM from an annotated corpus is simple.
  - **Supervised** learning: we have examples labelled with the right ‘answers’ (here, tags): no hidden variables in training.
- Training from unannotated corpus is trickier.
  - **Unsupervised** learning: we have no examples labelled with the right ‘answers’: all we see are outputs, state sequence is hidden.

Circularity

- If we know the state sequence, we can find the best λ.
  - E.g., use MLE:

- If we know λ, we can find the best state sequence.
  - use Viterbi

- But we don't know either!

Expectation-maximization (EM)

Essentially, a bootstrapping algorithm.

- Initialize parameters $\lambda^{(0)}$
- At each iteration k,
  - E-step: Compute **expected counts** using $\lambda^{(k-1)}$
  - M-step: Set $\lambda^{(k)}$ using MLE on the expected counts
- Repeat until λ doesn't change (or other stopping criterion).

Expected counts??

Counting transitions from $q^i \rightarrow q^j$:

- Real counts:
  - count 1 each time we see $q^i \rightarrow q^j$ in true tag sequence.
- Expected counts:
  - With current λ, compute probs of all possible tag sequences.
  - If sequence Q has probability p, count p for each $q^i \rightarrow q^j$ in Q.
  - Add up these fractional counts across all possible sequences.

Example

- Notionally, we compute expected counts as follows:

Possible sequence				Probability of sequence
$Q_1=$	q^1	q^1	q^1	p_1
$Q_2=$	q^1	q^2	q^1	p_2
$Q_3=$	q^1	q^1	q^2	p_3
$Q_4=$	q^1	q^2	q^2	p_4
Observs:	x	z	y	

Example

- Notionally, we compute expected counts as follows:

Possible sequence				Probability of sequence
Q_1=	q^1	q^1	q^1	p_1
Q_2=	q^1	q^2	q^1	p_2
Q_3=	q^1	q^1	q^2	p_3
Q_4=	q^1	q^2	q^2	p_4
Observs:	x	z	y	

$$\hat{C}(q1 \rightarrow q1) = 2p1 + p3$$

Forward-Backward algorithm

- As usual, avoid enumerating all possible sequences.
- **Forward-Backward** (Baum-Welch) algorithm computes expected counts using forward probabilities and **backward probabilities**:

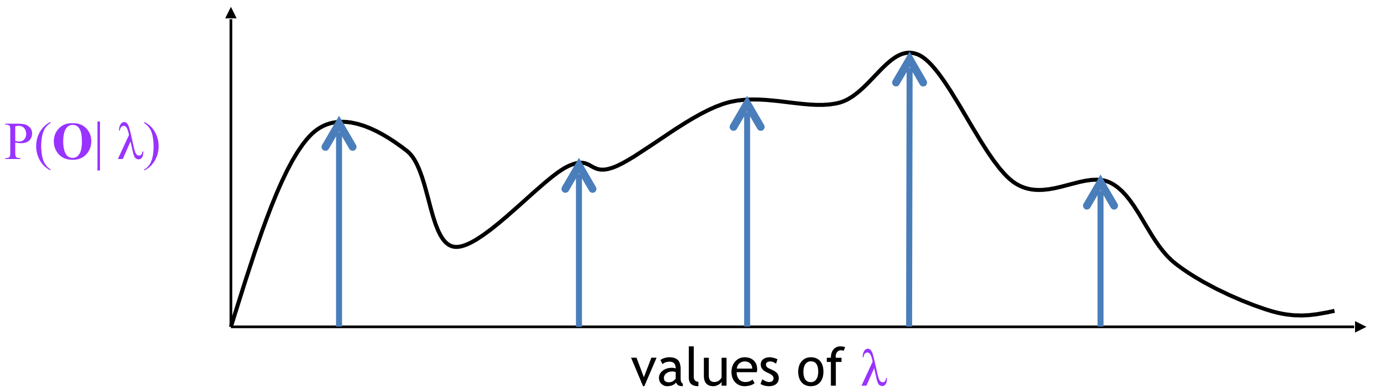
$$\beta(j,t) = P(qt = j,\ o_{t+1}, o_{t+2}, \ldots o_T \,|\, \lambda)$$

  – Details, see J&M 6.5

- EM idea is much more general: can use for many latent variable models.

Guarantees

- EM is guaranteed to find a **local** maximum of the likelihood.



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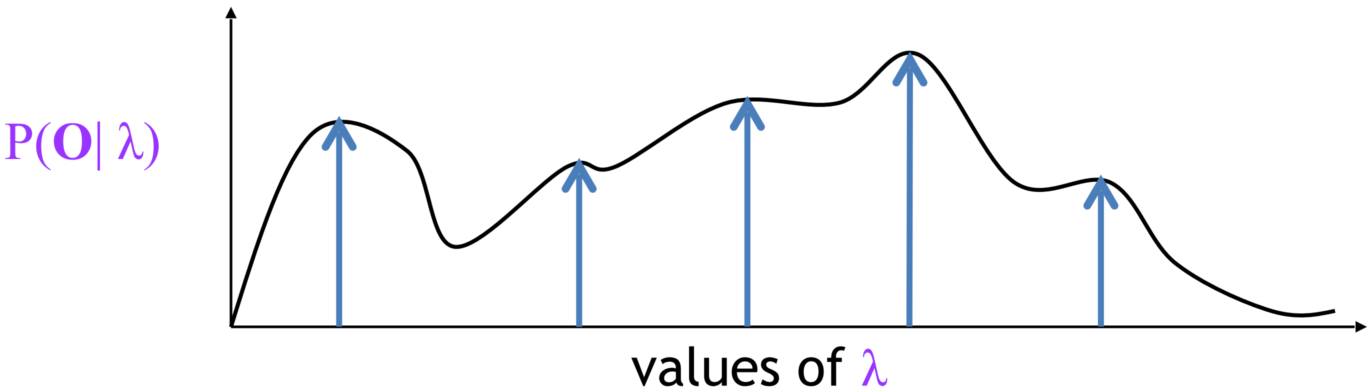

- Not guaranteed to find **global** maximum.
- Practical issues: initialization, random restarts, early stopping.

Summary

- HMM: a generative model of sentences using hidden state sequence
- Dynamic programming algorithms to compute
  - Best tag sequence given words (Viterbi algorithm)
  - Likelihood (forward algorithm)
  - Best parameters from unannotated corpus (forward-backward algorithm, an instance of EM)