Formal Modeling in Cognitive Science 1 (2005–2006)

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Solutions for Tutorial 9: Entropy; Mutual Information

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1. **Relationship between Entropy and Mutual Information**

   Let X and Y be two non-independent random variables. You know the entropies $H(X)$ and $H(Y)$ and the conditional entropies $H(X|Y)$ and $H(Y|X)$.

   (a) What is the mutual information of X with itself?
   Solution:
   $$I(X;X) = H(X) - H(X|X) = H(X)$$
   since $H(X|X) = 0$.

   (b) What is the joint entropy $H(X,Y)$, and what would it be if the random variables X and Y were independent?
   Solution: If X and Y are not independent, then:
   $$H(X,Y) = H(X) + H(Y|X)$$
   If X and Y are independent, then $H(Y|X) = H(Y)$:
   $$H(X,Y) = H(X) + H(Y)$$

   (c) Give an alternative expression for $I(X;Y)$ in terms of the joint entropy $H(X,Y)$ and the entropies $H(X)$ and $H(Y)$.
   Solution: We know that:
   $$I(X;Y) = H(X) - H(X|Y)$$
   We know that for joint entropy, the following holds:
   $$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$
   Therefore:
   $$H(X|Y) = H(X,Y) - H(Y)$$
   We can now compute mutual information as follows:
   $$\begin{aligned} I(X;Y) &= H(X) - H(X|Y) \\ &= H(X) - (H(X,Y) - H(Y)) \\ &= H(X) - H(X,Y) + H(Y) \end{aligned}$$

2. **Computing Entropy and Mutual Information**

   Let X and Y be a random variables over the sample space $S = \{a,b,c,d\}$. The joint distribution of these two random variables is as follows:

		x			
		a	b	c	d
	a	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$
y	b	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	0
	c	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{16}$	0
	d	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{16}$	0

(a) Write down the marginal distribution for X and compute the entropy $H(X)$.

Solution:

$$\begin{aligned} f(a) &= f(a,a)+f(b,a)+f(c,a)+f(d,a) \\ &= \tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}+\tfrac{1}{32} \\ &= \tfrac{1}{4} \end{aligned}$$

$$\begin{aligned} f(b) &= f(a,b)+f(b,b)+f(c,b)+f(d,b) \\ &= \tfrac{1}{16}+\tfrac{1}{8}+\tfrac{1}{32}+\tfrac{1}{32} \\ &= \tfrac{1}{4} \end{aligned}$$

$$\begin{aligned} f(c) &= f(a,c)+f(b,c)+f(c,c)+f(d,c) \\ &= \tfrac{1}{16}+\tfrac{1}{16}+\tfrac{1}{16}+\tfrac{1}{16} \\ &= \tfrac{1}{4} \end{aligned}$$

$$\begin{aligned} f(d) &= f(a,d)+f(b,d)+f(c,d)+f(d,d) \\ &= \tfrac{1}{4}+0+0+0 \\ &= \tfrac{1}{4} \end{aligned}$$

$$\begin{aligned} H(X) &= -\textstyle\sum_{x\in X} f(x)\log f(x) \\ &= -4\cdot\tfrac{1}{4}\log\tfrac{1}{4} \\ &= -\log\tfrac{1}{4} = 2 \end{aligned}$$

(b) Write down the marginal distribution for Y and compute the entropy $H(Y)$.

Solution: See question (2a).

(c) What is the joint entropy $H(X,Y)$ of the two random variables?

Solution: To calculate the joint entropy $H(X,Y)$ you need the conditional entropy $H(Y|X)$. For this you need the conditional distribution $f(y|x)$, which can be calculated as:

$$f(y|x) = \frac{f(y,x)}{f(x)}$$

for all values of X and Y. Alternatively, use the definition of joint entropy and calculate the joint entropy directly from the distribution $f(x,y)$:

$$H(X,Y) = -\sum_{x\in X}\sum_{y\in Y} f(x,y)\log f(x,y)$$

(d) What is the conditional entropy $H(Y|X)$?

Solution: See question (2c).

(e) What is the mutual information $I(X;Y)$ between the two random variables?

Solution: It is easy now to calculate the mutual information since we have worked out $H(X)$ and $H(X|Y)$:

$$I(X;Y) = H(X) - H(X|Y)$$

Alternatively, use the definition of mutual information:

$$I(X;Y) = \sum_{x\in X}\sum_{y\in Y} f(x,y)\log\frac{f(x,y)}{f(x)f(y)}$$

3. **Twenty Questions**

Consider a variant of the game Twenty Questions in which you have to guess which one of seven horses won a race. The probability distribution over winning horses is as follows:

horse	1	2	3	4	5	6	7
prob. of winning	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

(a) Assuming an optimal strategy, what is the minimum number of yes/no questions that you need to ask in order to find out which horse won?

Solution: The minimum number of questions is given by the entropy of the probability distribution:

$$\begin{aligned} H(X) &= -\sum_{x\in X} f(x)\log f(x) \\ &= \tfrac{1}{2}\log 4 + \tfrac{3}{8}\log 8 + \tfrac{1}{8}\log 16 = 1 + \tfrac{9}{8} + \tfrac{1}{2} = 2.625 \end{aligned}$$

(b) What is the expected number of questions if you adopt the following strategy: first ask about the horse with the highest probability, then about the horse with the second highest probability, etc.?

Solution:

horse	1	2	3	4	5	6	7
prob. of winning	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$
no. of questions	1	2	3	4	5	6	6

We can now compute the expectation of the number of questions that we must ask:

$$\begin{aligned} E(X) &= \sum_{x\in X} x f(x) \\ &= 1\cdot\tfrac{1}{4} + 2\cdot\tfrac{1}{4} + 3\cdot\tfrac{1}{8} + 4\cdot\tfrac{1}{8} + 5\cdot\tfrac{1}{8} + 6\cdot\tfrac{1}{16} + 6\cdot\tfrac{1}{16} = 3 \end{aligned}$$

This means that this strategy is not optimal, as $E(X) > H(X)$.