Formal Modeling in Cognitive Science 1 (2005–2006)

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Solutions for Tutorial 8: Expectation and Variance; Special Distributions

Week 9 (6-10 March, 2006)

1. Expectation and Variance

(a) For the discrete random variable *X* with the following probability distribution:

$$f(x) = \frac{|x-2|}{7}$$
 for $x = -1, 0, 1, 2, 3$

determine E(X) and var(X). Now assume the functions g(X) = 3X + 2 and $h(X) = X^2$ and determine E(g(X)) and E(h(X)).

Solution:

$$E(X) = \sum_{x} x \cdot f(x) = -1\frac{3}{7} + 0\frac{2}{7} + 1\frac{1}{7} + 2\frac{0}{7} + 3\frac{1}{7} = \frac{1}{7}$$

$$var(X) = \sum_{x} (x-\mu)^2 \cdot f(x) = (-1-\frac{1}{7})^2\frac{3}{7} + (0-\frac{1}{7})^2\frac{2}{7} + (1-\frac{1}{7})^2\frac{1}{7} + (2-\frac{1}{7})^2\frac{0}{7} + (3-\frac{1}{7})^2\frac{1}{7}$$

$$E(g(X)) = \sum_{x} 3x + 2 \cdot f(x) = -1\frac{3}{7} + 2\frac{2}{7} + 5\frac{1}{7} + 8\frac{0}{7} + 11\frac{1}{7} = \frac{16}{7}$$

$$E(h(X)) = \sum_{x} x^2 \cdot f(x) = (-1)^2\frac{3}{7} + 0^2\frac{2}{7} + 1^2\frac{1}{7} + 2^2\frac{0}{7} + 3^2\frac{1}{7} = \frac{13}{7}$$

(b) In Chebyshev's theorem, which form does the inequality take for k = 1, 2, 3, 4? Solution: The general form of Chebyshev's theorem is:

$$P(|x-\mu| < k\sigma) \ge 1 - \frac{1}{k^2}$$

So we get for k = 1, 2, 3, 4:

$$P(|x-\mu| < \sigma) \ge 0$$
$$P(|x-\mu| < 2\sigma) \ge \frac{3}{4}$$
$$P(|x-\mu| < 3\sigma) \ge \frac{8}{9}$$
$$P(|x-\mu| < 4\sigma) \ge \frac{15}{16}$$

2. Covariance

The *covariance* of two random variables *X* and *Y* with the joint distribution f(x, y) is defined as:

$$\operatorname{cov}(X,Y) = E((X - \mu_X)(Y - \mu_Y)) = \sum_{x} \sum_{y} (x - \mu_X)(y - \mu_Y) \cdot f(x,y)$$

where μ_X and μ_Y are the means of *X* and *Y*.

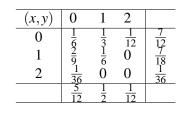
Assume that *X* and *Y* have the following joint distribution:

(x,y)	0	1	2
0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
1	$\overline{6}$ $\overline{2}$ $\overline{9}$	$\frac{1}{6}$	0
2	$\frac{1}{36}$	Ő	0

- (a) Compute the marginal distributions of *X* and *Y*.
- (b) Use the marginal distributions to compute μ_X and μ_Y .
- (c) Now compute the covariance of X and Y.

Solution:

(a) The marginal distributions are:



(b)

$$\mu_X = 0\frac{5}{12} + 1\frac{1}{2} + 2\frac{1}{12} = \frac{2}{3}$$
$$\mu_Y = 0\frac{7}{12} + 1\frac{7}{18} + 2\frac{1}{36} = \frac{4}{9}$$

(c)

$$\operatorname{cov}(X,Y) = \sum_{x} \sum_{y} (x - \frac{2}{3})(y - \frac{4}{9}) \cdot f(x,y) = -\frac{7}{54}$$

3. Special Distributions

(a) A scientist claims that 1 in 10 car accidents are due to driver fatigue. Using the formula for the binomial distribution, compute the probability that at most 3 of 5 accidents that happen on a given day are due to driver fatigue.
 Solution:

b(0;5,0.1) + b(1;5,0.1) + b(2;5,0.1) + b(3;5,0.1) = $\binom{5}{0}0.1^{0}0.9^{5} + \binom{5}{1}0.1^{1}0.9^{4} + \binom{5}{2}0.1^{2}0.9^{3} + \binom{5}{3}0.1^{3}0.9^{2} = 0.0086$

(b) In a reaction time experiment, the response latency in seconds is distributed according to the standard normal distribution. What is the probability that the reaction time is between 0 and 1 seconds?

Solution:

$$P(0 < X < 1) = \int_0^1 n(x;0,1) dx = \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0.3413$$

We can work this out by integrating, or because we know that for the normal distribution, $P(|x - \mu| < \sigma) = 0.6826$, hence for the standard normal distribution P(x < 1) = 0.6826/2 = 0.3413.