Formal Modeling in Cognitive Science 1 (2005–2006)

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Solutions for Tutorial 7: Random Variables and Probability Distributions

Week 8 (27 February–3 March, 2006)

1. **Probability Distributions and Probability Densities**

   (a) For each of the following, determine whether the given function can serve as the probability distribution for a random variable with the given range.

   i. $f(x) = \frac{x-2}{5}$ for $x = 1,2,3,4,5$
   ii. $f(x) = \frac{x^2}{30}$ for $x = 0,1,2,3,4$
   iii. $f(x) = \frac{1}{5}$ for $x = 0,1,2,3,4,5$

   Solution:

   i. No, because $f(1) = \frac{1-2}{5} = -\frac{1}{5} < 0$
   ii. Yes, because $\sum_{x=0}^{4} f(x) = \frac{0^2}{30} + \frac{1^2}{30} + \frac{2^2}{30} + \frac{3^2}{30} + \frac{4^2}{30} = 0 + \frac{1}{30} + \frac{4}{30} + \frac{9}{30} + \frac{16}{30} = \frac{30}{30} = 1$
   iii. No, because $\sum_{x=0}^{5} f(x) = 6 \cdot \frac{1}{5} = \frac{6}{5} > 1$

   (b) Find the cumulative distribution $F(x)$ for the random variable with the probability distribution:

   $$f(x) = \frac{x}{15} \text{ for } x = 1,2,3,4,5$$

   Solution:

   $$F(x) = \sum_{t \le x} f(t) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{15} & \text{for } 1 \le x < 2 \\ \frac{3}{15} & \text{for } 2 \le x < 3 \\ \frac{6}{15} & \text{for } 3 \le x < 4 \\ \frac{10}{15} & \text{for } 4 \le x < 5 \\ 1 & \text{for } x \ge 5 \end{cases}$$

   (c) The probability density function of the random variable X is given by:

   $$f(x) = \begin{cases} \frac{1}{8}(x+1) & \text{for } 2 \le x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

   i. Find $f(2 < X < 3)$.
   ii. Find the cumulative distribution $F(x)$.

   Solution:

   i. Compute the integral of $f(x)$ for $2 < X < 3$:

   $$\begin{aligned} P(2 \le X \le 3) &= \int_2^3 f(x)dx = \int_2^3 \frac{1}{8}(x+1)dx = \int_2^3 \frac{1}{8}x + \frac{1}{8}dx = \frac{1}{16}x^2 + \frac{1}{8}x\Big|_2^3 \\ &= (\frac{3^2}{16} + \frac{3}{8}) - (\frac{2^2}{16} + \frac{2}{8}) = \frac{7}{16} \end{aligned}$$

   ii. Compute the integral of $f(x)$ for $-\infty < X < x$:

   $$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt = \int_2^x \frac{1}{8}x + \frac{1}{8}dx = \frac{1}{16}x^2 + \frac{1}{8}x\Big|_2^x \\ &= (\frac{1}{16}x^2 + \frac{1}{8}x) - (\frac{2^2}{16} + \frac{2}{8}) = \frac{1}{16}x^2 + \frac{1}{8}x - \frac{1}{2} \end{aligned}$$

Therefore the cumulative distribution function is:

$$F(x) = \begin{cases} 0 & \text{for } x \leq 2 \\ \frac{1}{16}x^2 + \frac{1}{8}x - \frac{1}{2} & \text{for } 2 < x < 4 \\ 1 & \text{for } x \geq 4 \end{cases}$$

2. **Joint and Marginal Distributions**

Two psychometric tests are administered to a group of 10 experimental subjects. The first test is a standardized aptitude test used for university admissions. It returns a score between 0 and 100. Let X be the random variable denoting the aptitude test score. The second psychometric test is a memory test. It measures the digit span, i.e., the number of digits in a sequence that a subject is able to remember before they make a mistake. Let Y be the random variable that denotes the digit span, ranging from 1 to 5. The results of the experiment are given in the following table:

Subject	1	2	3	4	5	6	7	8	9	10
x	60	50	40	50	60	30	30	20	30	50
y	5	3	3	3	3	3	3	4	4	4

(a) Compute the distributions of X and Y.

(b) Compute the joint distribution of X and Y.

(c) Compute the marginal distributions of X and Y.

(d) Are X and Y independent?

(e) Compute the conditional distributions of X given $Y = 3$.

Solution:

(a)

x	20	30	40	50	60
$f(x)$	0.1	0.3	0.1	0.3	0.2
y	3	4	5		
$f(y)$	0.6	0.3	0.1		

(b)

(x,y)	20	30	40	50	60
3	0	0.2	0.1	0.2	0.1
4	0.1	0.1	0	0.1	0
5	0	0	0	0	0.1

(c)

(x,y)	20	30	40	50	60	$\sum_x f(x,y)$
3	0	0.2	0.1	0.2	0.1	0.6
4	0.1	0.1	0	0.1	0	0.3
5	0	0	0	0	0.1	0.1
$\sum_y f(x,y)$	0.1	0.3	0.1	0.3	0.2	

(d) No, for example $f(40,3) = 0.1$, but $\sum_y f(40,y) \cdot \sum_x f(x,3) = 0.1 \cdot 0.6 = 0.06$

(e)

x	20	30	40	50	60
$y = 3$	0	1/3	1/6	1/3	1/6